UNIMORE
UNIVERSITÀ DEGLI STUDI DI
MODENA E REGGIO EMILIA

DIPARTIMENTO DI SCIENZE FISICHE, INFORMATICHE E MATEMATICHE

Corso di Laurea Magistrale in Matematica

On Frobenius Groups

Relatore:
Prof. Cristina Acciarri

Candidato:
Alessia Gualano

Anno Accademico 2025–2026

Contents

Introduction

Finite group theory is one of the central branches of algebra. It developed during the nineteenth century from the study of permutations and algebraic equations, and it has since become a fundamental part of modern algebra. Its aim is to understand groups with finitely many elements and to describe the structural properties that distinguish them. Although the definition of a group is simple, finite groups can have a very rich and complex structure. One of the main strategies of the theory is therefore to study them through more accessible objects: special families of subgroups, actions on finite sets, and linear representations. These perspectives are closely connected. Subgroups often arise naturally as stabilizers of actions, and actions on finite sets allow one to study a group through the permutations it induces. In many cases, structural information about a group can be obtained precisely by combining these points of view.

A remarkable example of this interaction is provided by Frobenius groups. They are named after Ferdinand Georg Frobenius 1849–1917, a German mathematician whose work had a deep influence on the development of modern algebra. Frobenius was born in Charlottenburg, near Berlin, and studied at the universities of Göttingen and Berlin. He received his doctorate in 1870 under the supervision of Karl Weierstrass and, after holding a position at the Polytechnikum in Zurich, returned to Berlin, where he became one of the leading figures of the Berlin school of mathematics.

At the beginning of the twentieth century, the study of finite groups was still strongly connected with permutation groups. In this setting, a natural question was to understand how the behaviour of point stabilizers and their conjugates could determine the structure of the whole group. Frobenius' Theorem arises precisely from this point of view. More precisely, one considers a finite group G acting transitively, but not regularly, on a finite set Ω, under the assumption that no nonidentity element of G fixes more than one point of Ω. Under these hypotheses, Frobenius proved that the elements of G that do not fix any point of Ω, together with the identity element, form a normal subgroup of G.

One of the most striking aspects of Frobenius' Theorem lies in the contrast between the simplicity of its statement and the depth of the methods required to prove it. Although the theorem concerns the structure of a finite group, its proof uses character theory in an

essential way. In fact, no elementary proof avoiding character theory is known in the standard literature for the general case. Character theory arises from the study of linear representations. If a finite group is represented by linear transformations, one can associate to each representation its character, obtained by taking traces. In this way, representation-theoretic information is encoded in functions on the group. Frobenius was among the first to use this point of view systematically in the study of finite groups. Frobenius' Theorem shows the power of characters as a tool capable of revealing information about the internal structure of finite groups. For this reason, it remains one of the classical and most striking examples of the deep connection between representation theory and finite group theory.

In honour of the German mathematician, the structures that naturally arise from this specific interaction between permutation actions and subgroup structure are called *Frobenius groups*. In the modern formulation in terms of subgroups, the role of a point stabilizer is played by a proper nontrivial subgroup H of a finite group G. This subgroup has the property that it intersects each of its distinct conjugates only in the identity element. Such a subgroup is called a *Frobenius complement*. The remarkable feature of Frobenius groups is that this apparently local condition, involving only the subgroup H and its conjugates, imposes a strong restriction on the whole group. Indeed, Frobenius' Theorem 2.3.1 shows that there exists a normal subgroup N of G, called the *Frobenius kernel*, such that the group G decomposes as a semidirect product of the kernel and the complement. This decomposition also connects Frobenius groups with fixed-point-free actions. The complement H acts by conjugation on the kernel N, and no nonidentity element of H fixes a nontrivial element of N. For this reason, Frobenius groups provide a natural meeting point between subgroup structure, permutation actions and representation theory. The structural consequences of Frobenius' Theorem do not end with the existence of the kernel. In 1959, the mathematician John G. Thompson proved in his doctoral thesis that the Frobenius kernel is always nilpotent. Thompson's Theorem 4.2.5 is one of the deepest results in the theory of Frobenius groups: it shows that the kernel, whose existence is already forced by the action of the complement, has a highly constrained internal structure. Further refinements concern the nilpotency class of the kernel, the structure of Frobenius complements, and the irreducible representations of the whole group.

The aim of this thesis is to present the theory of Frobenius groups by following the main ideas that naturally arise from Frobenius' Theorem: the formulation in terms of subgroups and conjugates, the connection with permutation actions, the nilpotency of the kernel, and some further structural and representation-theoretic properties. The exposition begins with the preliminary material needed for the subsequent discussion, collected in Chapter 1. The first part is devoted to character theory and complex representations of finite groups, while the second part introduces group actions and some standard structural

notions concerning finite groups. Chapter 2 is devoted to the proof of Frobenius' Theorem. After presenting the specific tools needed for the argument, we state the theorem; for the original work, see [Fro01].

Theorem 2.3.1 (Frobenius). *Let G be a finite group and let $H < G$ be a Frobenius complement. Then the subset*

$$N = \{\, g \in G \mid g \text{ is not conjugate to any nonidentity element of } H \,\}$$

is a normal subgroup of G, $G = NH$, and $N \cap H = 1$.

The chapter also introduces the Frobenius partition, which will be studied again later in connection with the structural properties of Frobenius groups. Chapter 3 is devoted to the characterization of Frobenius groups in terms of actions. The first part studies fixed-point-free actions by automorphisms and shows how the semidirect product decomposition of a Frobenius group G can be interpreted through the action of the complement H on the kernel N. The chapter then turns to actions on finite sets and proves Theorem 3.2.3, which gives an equivalent characterization of Frobenius groups in terms of Frobenius actions and explains their interpretation as permutation groups. Chapter 4 focuses on the nilpotency of the Frobenius kernel. The proof of this main result relies on the theory of normal p-complements. Therefore, after recalling some of their basic properties, we introduce Thompson's criterion for the existence of a normal p-complement in a finite group, given in Theorem 4.2.2. This criterion plays a crucial role in establishing Theorem 4.2.5, which states that the Frobenius kernel of a Frobenius group is always nilpotent. Chapter 5 collects some further properties of Frobenius groups. The starting point is that, although Theorem 4.2.5 shows that the Frobenius kernel is always nilpotent, its nilpotency class may in general be arbitrarily large. This leads to the study of additional restrictions on the kernel and on the complement. In particular, the chapter discusses Burnside's Theorem 5.0.1 on the Sylow subgroups of Frobenius complements and some bounds on the nilpotency class of the kernel in terms of the prime divisors of the order of the complement. Finally, the chapter returns to representation theory by describing the irreducible representations of a Frobenius group over an algebraically closed field whose characteristic does not divide the order of the group.

Chapter 1

Preliminary concepts

In this chapter we collect definitions and basic results that will be used throughout the thesis and are needed to understand the subsequent arguments. Throughout the thesis, we use standard notation. For the results recalled in this chapter, we refer to [JL04, Isa94] for the material on complex representations and character theory, and to [Ros09, Isa94] for the background on group actions and finite group theory.

1.1 Characters and representations

Definition 1.1.1 (F-algebra)**.** Let F be a field. An *F-algebra* is an F-vector space A equipped with a multiplication map

$$A \times A \longrightarrow A, \qquad (a, b) \longmapsto ab,$$

which is *F-bilinear*. That is, for all $a, b, c \in A$ and all $\lambda \in F$,

$$(a + b)c = ac + bc, \qquad a(b + c) = ab + ac, \qquad (\lambda a)b = a(\lambda b) = \lambda(ab).$$

We say that A is:

- *associative* if $(ab)c = a(bc)$ for all $a, b, c \in A$.
- *commutative* if $ab = ba$ for all $a, b \in A$.
- *unital* if there exists an element $1_A \in A$ such that $1_A a = a1_A = a$ for all $a \in A$.

We now introduce a fundamental example of an F-algebra, which is associative and unital: the *group algebra* of a finite group.

Definition 1.1.2 (Group algebra)**.** Let F be a field and let G be a finite group. The *group algebra* FG is the F-vector space with basis $\{\, g \mid g \in G \,\}$. Thus every element of FG can be written uniquely as a finite formal sum

$$\alpha = \sum_{g \in G} a_g \, g \qquad where \quad a_g \in F.$$

Addition and scalar multiplication are defined coefficientwise. Moreover, the product in FG is defined as follows: for any

$$\alpha = \sum_{g \in G} a_g \, g \qquad \text{and} \qquad \beta = \sum_{h \in G} b_h \, h,$$

we set

$$\alpha\beta \;=\; \sum_{g \in G} \sum_{h \in G} a_g b_h \, (gh),$$

where gh denotes the product of g and h in the group G. With this multiplication, FG is an associative F-algebra. Moreover, it is unital, and its identity element is the identity 1_G of G.

For our purpose, we will mainly work over the field $\mathbb{C}$.

Definition 1.1.3 (Complex representation)**.** Let G be a finite group. A finite-dimensional complex representation of G is a group homomorphism

$$\rho : G \longrightarrow GL(V),$$

where V is a finite-dimensional vector space over $\mathbb{C}$ and $GL(V)$ denotes the group of invertible linear maps from V to itself. If $\dim_{\mathbb{C}} V = n$, then, after choosing a basis of V, the group $GL(V)$ is isomorphic to $GL_n(\mathbb{C})$. Thus, equivalently, one may regard ρ as a homomorphism

$$\rho : G \longrightarrow GL_n(\mathbb{C}).$$

The integer $n = \dim_{\mathbb{C}} V$ is called the degree of ρ.

Definition 1.1.4 (Character)**.** Let $\rho : G \to GL(V)$ be a complex representation. The character associated with ρ is the function

$$\chi_\rho : G \longrightarrow \mathbb{C}, \qquad \chi_\rho(g) = \mathrm{tr}(\rho(g)),$$

where tr denotes the trace of a linear operator. In this case, we say that ρ *affords* the character χ_ρ, or that χ_ρ is *afforded by* ρ. We write $\mathrm{Char}(G)$ for the set of complex characters of G.

If $\rho : G \to GL(V)$ is a complex representation with character χ_ρ, then

$$\chi_\rho(1) = \mathrm{tr}(\rho(1)) = \mathrm{tr}(\mathrm{Id}_V) = \dim_{\mathbb{C}} V.$$

Thus the value $\chi_\rho(1)$ is equal to the degree of the representation ρ. It is also called the *degree* of the character χ_ρ.

We now single out the representations which cannot be decomposed into smaller G-invariant pieces.

Definition 1.1.5 (Irreducible representation and irreducible character)**.** Let G be a finite group and let $\rho : G \to GL(V)$ be a finite-dimensional complex representation of G. We say that ρ is *irreducible* if V has no proper nonzero G-invariant subspaces, that is, the only G-invariant subspaces of V are 0 and V.

A character $\chi \in \mathrm{Char}(G)$ is called *irreducible* if it is afforded by an irreducible representation of G. We denote by $\mathrm{Irr}(G)$ the set of all irreducible characters of G.

One of the fundamental results concerning representations of finite groups is Maschke's theorem.

Theorem 1.1.6 (Maschke)**.** *Let G be a finite group and let $\rho : G \to GL(V)$ be a finite-dimensional complex representation of G. Then ρ is completely reducible. In other words, V can be written as a direct sum*

$$V = V_1 \oplus \cdots \oplus V_r,$$

where each V_i is an irreducible G-invariant subspace of V.

The first consequence of Maschke's Theorem 1.1.6 is that complex characters decompose as sums of irreducible characters.

Corollary 1.1.7. *Every complex character of G is a nonnegative integer linear combination of irreducible characters. More precisely, if $\chi \in \mathrm{Char}(G)$, then*

$$\chi = \sum_{\eta \in \mathrm{Irr}(G)} m_\eta \eta, \qquad m_\eta \in \mathbb{Z}_{\geq 0}.$$

The integer m_η is called the multiplicity with which η occurs in χ.

Proof. Let $\chi \in \mathrm{Char}(G)$, and let $\rho : G \to GL(V)$ be a representation affording χ. By Maschke's theorem 1.1.6, we may write

$$V = V_1 \oplus \cdots \oplus V_r,$$

where each V_i is an irreducible G-invariant subspace of V.

For each i, let χ_i be the character afforded by the representation

$$\rho_i : G \to GL(V_i), \qquad \rho_i(g) = \rho(g)|_{V_i}.$$

Then each χ_i is irreducible.

For each i, let $\mathcal{B}_i$ be a basis of V_i. Then $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$ is a basis of V. Since each V_i is G-invariant, with respect to this basis $\mathcal{B}$ the matrix of $\rho(g)$ is block diagonal, with blocks $\rho_1(g), \ldots, \rho_r(g)$. Hence $\chi(g) = \chi_1(g) + \cdots + \chi_r(g)$ for every $g \in G$, and so $\chi = \chi_1 + \cdots + \chi_r$. Grouping together equal irreducible characters, we obtain

$$\chi = \sum_{\eta \in \mathrm{Irr}(G)} m_\eta \eta, \qquad m_\eta \in \mathbb{Z}_{\geq 0}.$$

This completes the proof. □

A fundamental property of characters is that their values are constant on conjugacy classes. We use the notation $g^x = x^{-1}gx$ for the conjugate of an element $g \in G$ by an element $x \in G$. This motivates the introduction of the space of class functions.

Definition 1.1.8 (Class functions)**.** Let G be a finite group. A function $f : G \to \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes of G, that is,

$$f(g^x) = f(g) \qquad \text{for all } g, x \in G.$$

We denote by $\mathrm{cf}(G)$ the set of all class functions $G \to \mathbb{C}$. With pointwise addition and scalar multiplication, $\mathrm{cf}(G)$ is a complex vector space with dimension the number of conjugacy classes of G.

Every character is a class function. Indeed, if $\rho : G \to \mathrm{GL}(V)$ is a representation and $x, g \in G$, then

$$\rho(x^{-1}gx) = \rho(x)^{-1}\rho(g)\rho(x),$$

so $\rho(x^{-1}gx)$ is similar to $\rho(g)$. Since the trace is invariant under similarity, we have $\chi_\rho(x^{-1}gx) = \chi_\rho(g)$. Therefore

$$\mathrm{Irr}(G) \subseteq \mathrm{Char}(G) \subseteq \mathrm{cf}(G).$$

Definition 1.1.9 (Inner product on $\mathrm{cf}(G)$)**.** For $\alpha, \beta \in \mathrm{cf}(G)$, we define

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}.$$

This defines a Hermitian inner product on $\mathrm{cf}(G)$. Indeed, for every $\lambda \in \mathbb{C}$ and every $\alpha, \beta, \gamma \in \mathrm{cf}(G)$, we have

$$\langle \lambda\alpha + \beta, \gamma \rangle = \frac{1}{|G|} \sum_{g \in G} (\lambda\alpha(g) + \beta(g))\overline{\gamma(g)} = \lambda\langle \alpha, \gamma \rangle + \langle \beta, \gamma \rangle,$$

so the form is linear in the first variable. Moreover,

$$\langle \alpha, \lambda\beta + \gamma \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\lambda\beta(g) + \gamma(g)} = \overline{\lambda}\langle \alpha, \beta \rangle + \langle \alpha, \gamma \rangle,$$

so it is conjugate-linear in the second variable. Finally, $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$ and

$$\langle \alpha, \alpha \rangle = \frac{1}{|G|} \sum_{g \in G} |\alpha(g)|^2 \geq 0,$$

with equality if and only if $\alpha = 0$.

Irreducible characters play a central role in character theory. We now recall two of the basic results of character theory, which will be used repeatedly in what follows. Their proofs can be found, for instance, in [JL04]. We denote by $k(G)$ the number of conjugacy classes of G.

Theorem 1.1.10 (Orthonormality relations for irreducible characters)**.** *Let G be a finite group. If $\chi, \psi \in \mathrm{Irr}(G)$, then*

$$\langle \chi, \psi \rangle = \begin{cases} 1, & \text{if } \chi = \psi, \\ 0, & \text{if } \chi \neq \psi. \end{cases}$$

In particular, distinct irreducible characters are orthogonal.

Theorem 1.1.11. *The number of irreducible characters of G is equal to the number of conjugacy classes of G. In other words,*

$$|\mathrm{Irr}(G)| = k(G).$$

We now obtain the following consequence for the space $\mathrm{cf}(G)$.

Corollary 1.1.12. *The set $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathrm{cf}(G)$. In particular, every class function $\alpha \in \mathrm{cf}(G)$ can be written uniquely as*

$$\alpha = \sum_{\chi \in \mathrm{Irr}(G)} \langle \alpha, \chi \rangle \, \chi.$$

Proof. By Theorem 1.1.10, the elements of $\mathrm{Irr}(G)$ form an orthonormal set in $\mathrm{cf}(G)$. Moreover, by Theorem 1.1.11, the number of irreducible characters of G is equal to the number of conjugacy classes of G.

On the other hand, by Definition 1.1.8, the dimension of $\mathrm{cf}(G)$ is the number of conjugacy classes of G. Hence $\mathrm{Irr}(G)$ is an orthonormal set whose cardinality is equal to $\dim_{\mathbb{C}} \mathrm{cf}(G)$. Therefore $\mathrm{Irr}(G)$ is an orthonormal basis of $\mathrm{cf}(G)$.

Every class function $\alpha \in \mathrm{cf}(G)$ can be written uniquely as a linear combination of irreducible characters,

$$\alpha = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi.$$

Taking the inner product with a fixed $\psi \in \mathrm{Irr}(G)$, and using orthonormality, we obtain

$$\langle \alpha, \psi \rangle = \left\langle \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi, \psi \right\rangle = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \langle \chi, \psi \rangle = a_\psi.$$

Thus $a_\psi = \langle \alpha, \psi \rangle$ for every $\psi \in \mathrm{Irr}(G)$, and therefore

$$\alpha = \sum_{\chi \in \mathrm{Irr}(G)} \langle \alpha, \chi \rangle \, \chi.$$

□

We have seen that every character is a class function. We now give a criterion to determine when a class function is a character.

Corollary 1.1.13 (Character criterion)**.** *Let $\alpha \in \mathrm{cf}(G)$ be nonzero. Then α is a character of G if and only if*

$$\langle \alpha, \chi \rangle \in \mathbb{Z}_{\geq 0} \qquad \textit{for every } \chi \in \mathrm{Irr}(G).$$

Moreover, if $\chi, \psi \in \mathrm{Char}(G)$, then

$$\langle \chi, \psi \rangle \in \mathbb{Z}_{\geq 0}.$$

Proof. By Corollary 1.1.12, every class function $\alpha \in \mathrm{cf}(G)$ has a unique decomposition

$$\alpha = \sum_{\chi \in \mathrm{Irr}(G)} \langle \alpha, \chi \rangle \chi. \tag{1.1.1}$$

If α is a character, then by Corollary 1.1.7 it is a nonnegative integer linear combination of irreducible characters. By uniqueness of the decomposition (1.1.1), we have

$$\langle \alpha, \chi \rangle \in \mathbb{Z}_{\geq 0} \qquad \text{for every } \chi \in \mathrm{Irr}(G).$$

Conversely, if $\langle \alpha, \chi \rangle \in \mathbb{Z}_{\geq 0}$ for every $\chi \in \mathrm{Irr}(G)$, then the expression (1.1.1) shows that α is a nonnegative integer linear combination of irreducible characters. Hence α is a character.

Now let $\chi, \psi \in \mathrm{Char}(G)$. By the argument above, we may write

$$\chi = \sum_{\eta \in \mathrm{Irr}(G)} a_\eta \eta, \qquad \psi = \sum_{\eta \in \mathrm{Irr}(G)} b_\eta \eta,$$

with $a_\eta, b_\eta \in \mathbb{Z}_{\geq 0}$. Using the orthonormality of irreducible characters Theorem 1.1.10, we obtain

$$\langle \chi, \psi \rangle = \sum_{\eta \in \mathrm{Irr}(G)} a_\eta b_\eta \in \mathbb{Z}_{\geq 0},$$

and this completes the proof. □

We now introduce the regular character and collect some basic properties that will be used later.

Example 1.1.14 (The regular character)**.** Let G be a finite group and consider the complex vector space $\mathbb{C}G$ with basis G. For each $a \in G$ we define a $\mathbb{C}$-linear map

$$T_a : \mathbb{C}G \longrightarrow \mathbb{C}G$$

by prescribing its action on the basis elements:

$$T_a(x) = ax \qquad (x \in G),$$

and extending by $\mathbb{C}$-linearity to all of $\mathbb{C}G$. Since $T_{ab}(x) = abx = T_a \circ T_b(x)$ for all $a, b, x \in G$, the assignment

$$\rho_{\mathrm{reg}} : G \longrightarrow \mathrm{GL}(\mathbb{C}G), \qquad \rho_{\mathrm{reg}}(a) = T_a,$$

is a group homomorphism. We call ρ_{reg} the *(left) regular representation* of G. The character associated with ρ_{reg} is denoted by χ_{reg} and is called the *regular character* of G.

Proposition 1.1.15. *The regular character of a finite group G satisfies*

$$\chi_{\mathrm{reg}}(1) = |G| \qquad \textit{and} \qquad \chi_{\mathrm{reg}}(g) = 0 \ \textit{for all } g \in G \setminus \{1\}.$$

Proof. We use the basis G of $\mathbb{C}G$. For $x \in G$, the linear map $\rho_{\mathrm{reg}}(x)$ permutes the basis elements by $g \mapsto xg$. Hence the trace equals the number of fixed basis vectors. If $x = 1$, every basis element is fixed, so the trace is $|G|$. If $x \neq 1$, then $xg \neq g$ for all $g \in G$, so there are no fixed basis vectors and the trace is 0. □

Proposition 1.1.16. *For the regular character χ_{reg} we have the decomposition*

$$\chi_{\mathrm{reg}} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\,\chi.$$

Equivalently, for every $\chi \in \mathrm{Irr}(G)$,

$$\langle \chi_{\mathrm{reg}}, \chi \rangle = \chi(1).$$

Proof. By Corollary 1.1.12, it is enough to compute the coefficients $\langle \chi_{\mathrm{reg}}, \chi \rangle$ for any $\chi \in \mathrm{Irr}(G)$. Using $\chi_{\mathrm{reg}}(1) = |G|$ and $\chi_{\mathrm{reg}}(g) = 0$ for $g \neq 1$, we get

$$\langle \chi_{\mathrm{reg}}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{reg}}(g)\overline{\chi(g)} = \frac{1}{|G|}\,|G|\,\overline{\chi(1)} = \chi(1),$$

since $\chi(1)$ is a nonnegative integer. □

Corollary 1.1.17. *For every finite group G, we have*

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

Proof. We consider the value of the regular character at the identity element. By Proposition 1.1.16, we have

$$\chi_{\mathrm{reg}}(1) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi(1) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

On the other hand, by Proposition 1.1.15, $\chi_{\mathrm{reg}}(1) = |G|$. This completes the proof. □

Character-theoretic information naturally determines normal subgroups of the group.

Lemma 1.1.18. *Let G be a finite group and let $\chi \in \mathrm{Char}(G)$. Define*

$$\ker(\chi) = \{\, g \in G \mid \chi(g) = \chi(1) \,\}, \qquad Z(\chi) = \{\, g \in G \mid |\chi(g)| = \chi(1) \,\}.$$

Then $\ker(\chi)$ and $Z(\chi)$ are normal subgroups of G, and

$$Z(\chi)/\ker(\chi) \;\leq\; \mathrm{Z}\big(G/\ker(\chi)\big).$$

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional complex representation affording χ, so that $\chi(g) = \mathrm{tr}(\rho(g))$ for all $g \in G$, and let $n = \chi(1) = \dim_{\mathbb{C}}(V)$.

We show that $\ker(\chi) = \ker(\rho)$. Since $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism, its kernel $\ker(\rho)$ is a normal subgroup of G. Therefore, once we prove $\ker(\chi) = \ker(\rho)$, it will follow immediately that $\ker(\chi)$ is normal in G.
If $g \in \ker(\rho)$ then $\rho(g) = \mathrm{Id}_V$, hence $\chi(g) = \mathrm{tr}(\mathrm{Id}_V) = n$, so $g \in \ker(\chi)$. Conversely, assume that $\chi(g) = n$. Over $\mathbb{C}$, the matrix $\rho(g)$ is diagonalizable and all its eigenvalues are roots of unity, hence have absolute value 1. Write these eigenvalues as $\varepsilon_1, \ldots, \varepsilon_n$ counted with multiplicity. Then $\chi(g) = \mathrm{tr}(\rho(g)) = \varepsilon_1 + \cdots + \varepsilon_n$. By the triangle inequality,

$$|\chi(g)| = |\varepsilon_1 + \cdots + \varepsilon_n| \leq |\varepsilon_1| + \cdots + |\varepsilon_n| = n.$$

Since $\chi(g) = n$, equality holds and therefore all ε_i must be equal. Thus $\varepsilon_1 = \cdots = \varepsilon_n = \varepsilon$ for some complex number ε with $|\varepsilon| = 1$, and

$$\chi(g) = \varepsilon_1 + \cdots + \varepsilon_n = n\varepsilon = n.$$

Hence $\varepsilon = 1$ and so $\rho(g) = \mathrm{Id}_V$, i.e. $g \in \ker(\rho)$. We have shown that $\ker(\chi) = \ker(\rho)$, and in particular $\ker(\chi)$ is normal in G.

Next we describe $Z(\chi)$. Assume that $g \in Z(\chi)$, that is, $|\chi(g)| = n$. With the same notation as above, let $\varepsilon_1, \ldots, \varepsilon_n$ be the eigenvalues of $\rho(g)$. Then

$$|\varepsilon_1 + \cdots + \varepsilon_n| = n \qquad \text{and} \qquad |\varepsilon_i| = 1 \quad \text{for all } i.$$

Again, equality in the triangle inequality implies that all the eigenvalues are equal. Hence $\varepsilon_1 = \cdots = \varepsilon_n = \varepsilon$ for some $\varepsilon \in \mathbb{C}$ with $|\varepsilon| = 1$. Therefore $\rho(g) = \varepsilon\,\mathrm{Id}_V$.
Conversely, if $\rho(g) = \varepsilon\,\mathrm{Id}_V$ with $|\varepsilon| = 1$, then $\chi(g) = \mathrm{tr}(\rho(g)) = n\varepsilon$, and hence $|\chi(g)| = n$. Thus

$$Z(\chi) = \{g \in G \mid \rho(g) = \varepsilon\,\mathrm{Id}_V \text{ for some } \varepsilon \in \mathbb{C}^\times\}.$$

Moreover, since $\ker(\chi) = \ker(\rho)$, we have $\ker(\chi) \leq Z(\chi)$. Indeed, if $g \in \ker(\chi)$, then $\rho(g) = \mathrm{Id}_V$, so in particular $\rho(g)$ is scalar.

Now set

$$K = \ker(\chi) = \ker(\rho).$$

By the first isomorphism theorem, the map

$$G/K \longrightarrow \rho(G), \qquad gK \longmapsto \rho(g),$$

is an isomorphism.

Let

$$S = \{\varepsilon\,\mathrm{Id}_V \mid \varepsilon \in \mathbb{C}^\times\} \cap \rho(G).$$

By the description above, we have $\rho(Z(\chi)) = S$. Moreover, under the isomorphism $G/K \simeq \rho(G)$, the subgroup $Z(\chi)/K$ corresponds exactly to S.

Since scalar operators commute with every linear operator, S is contained in the center of $\rho(G)$:

$$S \leq \mathrm{Z}(\rho(G)).$$

It follows, through the isomorphism $G/K \simeq \rho(G)$, that $Z(\chi)/K \leq \mathrm{Z}(G/K)$. Since $K = \ker(\chi)$, this is precisely

$$Z(\chi)/\ker(\chi) \leq \mathrm{Z}(G/\ker(\chi)).$$

This completes the proof. □

Corollary 1.1.19. *The intersection of the kernels of the irreducible characters of G is trivial:*

$$\bigcap_{\chi \in \mathrm{Irr}(G)} \ker(\chi) = 1.$$

Proof. Let $g \in \bigcap_{\chi \in \mathrm{Irr}(G)} \ker(\chi)$. Then $\chi(g) = \chi(1)$ for every $\chi \in \mathrm{Irr}(G)$. Consider the regular character χ_{reg}. On the one hand, by Proposition 1.1.16

$$\chi_{\mathrm{reg}} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\,\chi.$$

It follows that for $g \in \bigcap_{\chi \in \mathrm{Irr}(G)} \ker(\chi)$ we have

$$\chi_{\mathrm{reg}}(g) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi(g) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \neq 0.$$

On the other hand, the regular character satisfies $\chi_{\mathrm{reg}}(x) = 0$ for all $x \neq 1$, hence we must have $g = 1$. □

1.2 Group actions and structural results

In this section we recall some basic notions on group actions and some structural results on finite groups. We first consider actions on arbitrary sets, and then we specialize to actions on groups by automorphisms.

Let Ω be a nonempty set. A *permutation group* on Ω is a subgroup $G \leq \mathrm{Sym}(\Omega)$, where $\mathrm{Sym}(\Omega)$ denotes the group of all permutations of Ω. Equivalently, one may view G as a group acting faithfully on Ω.

Suppose now that a group G acts on a set Ω. For $\alpha \in \Omega$, the *orbit* of α is the set

$$\alpha^G = \{g \cdot \alpha \mid g \in G\},$$

while the *stabilizer* of α is

$$\mathrm{Stab}_G(\alpha) = \{g \in G \mid g \cdot \alpha = \alpha\}.$$

The action is said to be *transitive*, or equivalently G is said to *act transitively on* Ω, if for every $\alpha, \beta \in \Omega$ there exists $g \in G$ such that $g \cdot \alpha = \beta$. Equivalently, $\alpha^G = \Omega$ for every $\alpha \in \Omega$.

If $g \in G$, the set of fixed points of g is

$$\mathrm{Fix}_\Omega(g) = \{\alpha \in \Omega \mid g \cdot \alpha = \alpha\}.$$

We will also use the orbit-stabilizer theorem.

Theorem 1.2.1 (Orbit-stabilizer theorem)**.** *If a finite group G acts on a finite set Ω, and if $\alpha \in \Omega$, then*

$$|\alpha^G| = [G : \mathrm{Stab}_G(\alpha)] = \frac{|G|}{|\mathrm{Stab}_G(\alpha)|}.$$

In particular, if the action is transitive, then

$$|\Omega| = [G : \mathrm{Stab}_G(\alpha)]$$

for every $\alpha \in \Omega$.

Assume that G acts transitively on Ω, and fix $\alpha \in \Omega$. Then the map

$$\{g\,\mathrm{Stab}_G(\alpha) | g \in G\} \longrightarrow \Omega, \qquad g\,\mathrm{Stab}_G(\alpha) \longmapsto g \cdot \alpha,$$

is a well-defined bijection. Thus, whenever G acts transitively on Ω, the action can be studied as the natural action of G on the set of left cosets of a point stabilizer.

We now turn to actions on groups. In this setting, we will mainly be interested in actions which preserve the group structure, namely actions by automorphisms.

Let A and N be groups. An *action of A on N by automorphisms* is a group homomorphism

$$\varphi : A \longrightarrow \mathrm{Aut}(N).$$

For $a \in A$, we write

$$\mathrm{Fix}_N(a) = \mathrm{Fix}_N(\varphi(a)) = \{\, n \in N \mid \varphi(a)(n) = n \,\}.$$

A fundamental example of an action by automorphisms is given by conjugation. Let G be a group, let $N \lhd G$, and let $H \leq G$. Since N is normal in G, for every $h \in H$ the map

$$c_h : N \longrightarrow N, \qquad c_h(n) = hnh^{-1},$$

is an automorphism of N with inverse $c_{h^{-1}}$. Hence we obtain a group homomorphism

$$\varphi : H \longrightarrow \mathrm{Aut}(N), \qquad \varphi(h) = c_h.$$

Definition 1.2.2 (Internal semidirect product)**.** Let G be a group, let $N \lhd G$, and let $A \leq G$. We say that G is the *internal semidirect product* of N by A if $G = NA$ and $N \cap A = 1$. In this case, every element of G can be written uniquely in the form na, with $n \in N$ and $a \in A$, and we write

$$G = N \rtimes A.$$

In this situation, the subgroup A acts on N by conjugation.

We now introduce a special class of automorphisms.

Definition 1.2.3 (Fixed-point-free automorphism)**.** Let N be a group and let $\sigma \in \mathrm{Aut}(N)$. The set of fixed points of σ is $\mathrm{Fix}_N(\sigma) = \{\, n \in N \mid \sigma(n) = n \,\}$. We say that σ is *fixed-point-free* if

$$\mathrm{Fix}_N(\sigma) = \{1\}.$$

We now extend this notion to actions by automorphisms, which will play an important role throughout the thesis.

Definition 1.2.4 (Fixed-point-free action by automorphisms)**.** Let A and N be groups, and let $\varphi : A \longrightarrow \mathrm{Aut}(N)$ be an action of A on N by automorphisms. The action is called *fixed-point-free* if, for every $a \in A \setminus \{1\}$, one has

$$\mathrm{Fix}_N(a) = \{1\}.$$

We shall also use the following elementary properties of fixed-point-free automorphisms; see [Hup25, Theorem 8.9(b),(c)].

Proposition 1.2.5. *Let N be a finite group, and let $\sigma \in \mathrm{Aut}(N)$ be a fixed-point-free automorphism. Then the following hold.*

1. *The map $\mu : N \longrightarrow N, \quad \mu(n) = n^{-1}\sigma(n)$, is a bijection from N onto itself.*
2. *If n and $\sigma(n)$ are conjugate in N, then $n = 1$.*

Proof. We first prove that μ is injective. Let $n_1, n_2 \in N$, and suppose that $\mu(n_1) = \mu(n_2)$. Then $n_1^{-1}\sigma(n_1) = n_2^{-1}\sigma(n_2)$. It follows that

$$n_2 n_1^{-1} = \sigma(n_2)\sigma(n_1)^{-1} = \sigma(n_2 n_1^{-1}).$$

Thus $n_2 n_1^{-1}$ is fixed by σ. Since σ is fixed-point-free, we get $n_2 n_1^{-1} = 1$, and hence $n_1 = n_2$. Therefore μ is injective. Since N is finite, μ is also surjective. Therefore every element of N can be written in the form $n^{-1}\sigma(n)$ for some $n \in N$. Equivalently, $N = \{\, n^{-1}\sigma(n) \mid n \in N \,\}$.

We now prove the second statement. Suppose that n and $\sigma(n)$ are conjugate in N. Then there exists $x \in N$ such that

$$\sigma(n) = x^{-1}nx.$$

By the first part, the map μ is surjective. Hence there exists $y \in N$ such that $x = \mu(y) = y^{-1}\sigma(y)$. Therefore, we obtain

$$\sigma(n) = \big(y^{-1}\sigma(y)\big)^{-1} n y^{-1}\sigma(y) = \sigma(y)^{-1} y n y^{-1}\sigma(y).$$

Hence

$$\sigma(yny^{-1}) = \sigma(y)\sigma(n)\sigma(y)^{-1} = yny^{-1}.$$

Thus yny^{-1} is fixed by σ. Since σ is fixed-point-free, we get $yny^{-1} = 1$, and therefore $n = 1$. □

Remark 1.2.6. Let H act on N by conjugation, that is, $h \cdot n = hnh^{-1}$ for $h \in H$ and $n \in N$. In this situation, fixed points can be described in terms of centralizers.

For a fixed element $h \in H$, the set of points of N fixed by h is

$$\mathrm{Fix}_N(h) = \{\, n \in N \mid hnh^{-1} = n \,\}.$$

Since $hnh^{-1} = n$ is equivalent to $hn = nh$, this set is precisely the centralizer of h in N:

$$\mathrm{Fix}_N(h) = C_N(h).$$

Dually, for a fixed element $n \in N$, the stabilizer of n in the action of H on N is

$$\mathrm{Stab}_H(n) = \{\, h \in H \mid hnh^{-1} = n \,\}.$$

Again, $hnh^{-1} = n$ is equivalent to $hn = nh$, and therefore

$$\mathrm{Stab}_H(n) = C_H(n).$$

In particular, the conjugation action of H on N is fixed-point-free if and only if

$$C_N(h) = 1 \qquad \text{for every } h \in H \setminus \{1\}.$$

Equivalently,

$$C_H(n) = 1 \qquad \text{for every } n \in N \setminus \{1\}.$$

We now recall some standard results on coprime actions, which will be used to study the behaviour of fixed points under quotients.

Definition 1.2.7 (Coprime action)**.** Let A and G be finite groups, and suppose that A acts on G by automorphisms. The action is called *coprime* if $\big(|A|, |G|\big) = 1$.

For an action of A on G, we write g^a for the image of g under the automorphism induced by a. We denote by

$$C_G(A) = \{\, g \in G \mid g^a = g \text{ for every } a \in A \,\}$$

the subgroup of elements of G fixed by every element of A.

The following result describes the behaviour of cosets under coprime actions. A proof can be found in [Isa08, Theorem 3.27].

Theorem 1.2.8. *Let A act by automorphisms on a finite group G, and set $C = C_G(A)$. Let $H \leq G$ be an A-invariant subgroup. Assume that $\big(|A|, |H|\big) = 1$ and that either A or H is solvable. Then the A-invariant left cosets of H in G are exactly the left cosets of H which contain an element of C.*

Equivalently, a left coset gH is A-invariant if and only if

$$gH \cap C_G(A) \neq \emptyset.$$

The same statement holds for right cosets.

We will mainly use the following consequence; see [Isa08, Corollary 3.28]. It states that, for coprime actions, fixed points in a quotient are induced by fixed points in the original group.

Corollary 1.2.9. *Let A act by automorphisms on a finite group G, and let $N \triangleleft G$ be an A-invariant normal subgroup. Assume that $\big(|A|, |G|\big) = 1$ and that either A or N is*

solvable. Then

$$C_{G/N}(A) = C_G(A)N/N.$$

Proof. Since N is A-invariant, the action of A on G induces an action on the quotient G/N, given by

$$(gN)^a = g^a N$$

for every $g \in G$ and every $a \in A$.

Let $gN \in G/N$. The coset gN is fixed by every element of A if and only if

$$(gN)^a = gN$$

for every $a \in A$. Equivalently, $g^a N = gN$ for every $a \in A$, which means precisely that gN is an A-invariant coset.

By Theorem 1.2.8, applied with $H = N$, the A-invariant cosets of N in G are exactly those containing an element of $C_G(A)$. Hence gN is fixed by A if and only if there exists $c \in C_G(A)$ such that

$$gN = cN.$$

Therefore the fixed points of A on G/N are exactly the cosets of the form cN, with $c \in C_G(A)$, and so

$$C_{G/N}(A) = \{\, cN \mid c \in C_G(A) \,\} = C_G(A)N/N.$$

□

We now recall some basic structural notions on finite groups that will be used later.

We first fix the notation for commutators. For $x, y \in G$, we write

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y.$$

The subgroup of G generated by all commutators is denoted by $G' = [G, G]$ and is called the (first) derived subgroup of G. It is a characteristic subgroup of G.

The derived series of G is the descending sequence of subgroups defined as follows:

$$G^{(0)} = G, \qquad G^{(i+1)} = (G^{(i)})' \quad \text{for every } i \geq 0.$$

Definition 1.2.10 (Solvable group)**.** A group G is called *solvable* if its derived series eventually reaches the trivial subgroup, that is, if there exists an integer $n \geq 0$ such that

$$G^{(n)} = 1.$$

Intuitively, solvable groups are groups that can be constructed from abelian groups by successive extensions. In particular, every abelian group is solvable, since

$$G' = [G, G] = 1.$$

Nilpotency is a stronger condition than solvability. We first introduce the main notion needed to define it.

The lower central series of a group G is the descending sequence of subgroups defined inductively by:

$$\gamma_1(G) = G, \qquad \gamma_{i+1}(G) = [\gamma_i(G), G] \quad \text{for every } i \geq 1.$$

Thus

$$\gamma_2(G) = [G, G] = G',$$

and each subsequent term measures how far the previous one is from being central in G.

Definition 1.2.11 (Nilpotent group)**.** A group G is called *nilpotent* if there exists an integer $c \geq 1$ such that

$$\gamma_{c+1}(G) = 1.$$

The least such integer c is called the nilpotency class of G, and is denoted by $cl(G)$. In this case, we say that G is nilpotent of class c.

In particular, a group has nilpotency class 1 if and only if it is abelian: indeed,

$$\gamma_2(G) = [G, G] = 1$$

is equivalent to G being abelian.

For finite groups, nilpotency admits an equivalent characterization in terms of Sylow subgroups: a finite group is nilpotent if and only if all its Sylow subgroups are normal. Equivalently, a finite nilpotent group is the direct product of its Sylow subgroups; see, for instance, [Hup25, Theorems I.9.5 and III.2.3]. We now introduce another subgroup associated with a finite group.

Definition 1.2.12. Let G be a finite group. The nilpotent residual of G is the subgroup

$$\gamma_\infty(G) = \min\{\, K \lhd G \mid G/K \text{ is nilpotent} \,\}.$$

Intuitively, $\gamma_\infty(G)$ measures the obstruction preventing G from being nilpotent.

Lemma 1.2.13. *Let G be a finite group. Then the nilpotent residual $\gamma_\infty(G)$ is characteristic in G.*

Proof. Let $\alpha \in \mathrm{Aut}(G)$. We show that $\alpha(\gamma_\infty(G)) = \gamma_\infty(G)$.

Set $R = \gamma_\infty(G)$. By definition, $R \triangleleft G$ and G/R is nilpotent. First observe that $\alpha(R) \triangleleft G$. Moreover, the automorphism α induces an isomorphism

$$G/R \cong G/\alpha(R),$$

given by $gR \mapsto \alpha(g)\alpha(R)$. Since nilpotency is preserved under isomorphisms, $G/\alpha(R)$ is nilpotent.

Therefore $\alpha(R)$ is a normal subgroup of G whose quotient is nilpotent. By the minimality of $R = \gamma_\infty(G)$, we obtain $R \leq \alpha(R)$. Applying the same argument to α^{-1}, we also get $R \leq \alpha^{-1}(R)$. Applying α to this inclusion gives $\alpha(R) \leq R$. Thus $\alpha(R) = R$. Since this holds for every $\alpha \in \mathrm{Aut}(G)$, we conclude that $\gamma_\infty(G)$ is characteristic in G. □

We shall also use the following classical result; see, for instance, [Isa08, Corollary 1.9].

Theorem 1.2.14 (Cauchy's theorem)**.** *Let G be a finite group, and let p be a prime dividing $|G|$. Then G contains an element of order p.*

We next introduce a standard normal subgroup associated with a prime p.

Definition 1.2.15 (p'-core)**.** Let G be a finite group, and let p be a prime. The p'-core of G, denoted by $O_{p'}(G)$, is the largest normal subgroup of G whose order is not divisible by p. Equivalently,

$$O_{p'}(G) = \langle\, K \triangleleft G \mid p \nmid |K| \,\rangle.$$

Definition 1.2.16 (Minimal normal subgroup)**.** Let G be a group. A subgroup $U \triangleleft G$ is called a *minimal normal subgroup* of G if $U \neq 1$ and the only normal subgroups of G contained in U are 1 and U itself.

Proposition 1.2.17. *Let U be a solvable minimal normal subgroup of a finite group G. Then U is elementary abelian. Equivalently,*

$$U \cong C_p^n$$

for some prime p and some integer $n \geq 1$.

Proof. Since the derived subgroup U' is characteristic in U and U is normal in G, we have that U' is normal in G. By the minimality of U, either $U' = 1$ or $U' = U$. The second possibility cannot occur because U is solvable. Hence $U' = 1$, and so U is abelian.

Now let p be a prime dividing $|U|$, and set $S = \{x \in U \mid x^p = 1\}$. Since U is abelian, S is a subgroup of U. Moreover, S is characteristic in U, because every automorphism

of U preserves the orders of elements. Therefore S is normal in G. Moreover, $S \neq 1$ by Cauchy's Theorem 1.2.14. Hence, by the minimality of U, we have $S = U$. Thus every element of U has order dividing p, and consequently

$$U \cong C_p^n$$

for some integer $n \geq 1$, as desired. □

We now recall the notion of a complement. Let G be a group and let $N \lhd G$. A subgroup $H \leq G$ is called a *complement* of N in G if

$$G = NH \qquad \text{and} \qquad N \cap H = 1.$$

In this case, G is the internal semidirect product of N by H. If, in addition, $H \lhd G$, then H is called a *normal complement* of N in G. The following result is the Schur–Zassenhaus Theorem. It gives the existence of complements for normal Hall subgroups and gives conditions under which such complements are conjugate; see [Isa08, Section 3B].

Theorem 1.2.18 (Schur–Zassenhaus)**.** *Let G be a finite group and let $N \lhd G$. Assume that $\gcd(|N|, |G/N|) = 1$. Then N has a complement in G, that is, there exists a subgroup $H \leq G$ such that $G = NH$ and $N \cap H = 1$. Moreover, if either N or G/N is solvable, then all complements of N in G are conjugate in G. In other words, if H_1 and H_2 are subgroups of G such that*

$$G = NH_1 = NH_2 \qquad \textit{and} \qquad N \cap H_1 = N \cap H_2 = 1,$$

then there exists an element $g \in G$ such that $H_2 = H_1^g$.

Chapter 2

The Frobenius Theorem

Our objective is twofold: to develop the main tools needed for Frobenius' Theorem and to establish the theorem. The result describes a remarkable situation in which a finite group G contains a proper subgroup H whose conjugates intersect trivially, and shows that this condition forces a strong structural decomposition of G. More precisely, under this hypothesis one can identify a distinguished normal subgroup $N \trianglelefteq G$ such that $G = NH$ and $N \cap H = 1$. The proof of this result relies on character theory, and in particular on the interaction between characters of G and characters of its subgroups. Frobenius' theorem is a classical example of a group-theoretic result whose known proofs essentially use character theory. The material in this chapter follows the presentation in [Isa94, Chapter 15].

2.1 Induced characters and Frobenius reciprocity

In this section we introduce the operations of restriction and induction of class functions, which allow us to relate functions defined on a subgroup H to functions on the whole group G. A central result is Frobenius reciprocity, which establishes a precise compatibility between these two operations and will play a key role in the proof of Frobenius' Theorem.

A first guiding question is the following: given a finite group G and a subgroup H of G, how are the characters of G related to the characters of H?

Definition 2.1.1 (Restriction of class functions)**.** Let $H \leq G$ and let $\varphi \in \mathrm{cf}(G)$. The *restriction* of φ to H is the function

$$\varphi_H : H \longrightarrow \mathbb{C}, \qquad \varphi_H(h) = \varphi(h) \quad \forall h \in H.$$

Observe that the restriction of a class function is again a class function. Indeed, if

$x, h \in H$, then $x^{-1}hx$ is also a conjugate of h in G, and hence

$$\varphi_H(x^{-1}hx) = \varphi(x^{-1}hx) = \varphi(h) = \varphi_H(h).$$

A stronger property holds for characters.

Proposition 2.1.2. *Let $H \leq G$ and let $\chi \in \mathrm{Char}(G)$. Then $\chi_H \in \mathrm{Char}(H)$.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a complex representation of G affording χ, so that $\chi(g) = \mathrm{tr}(\rho(g))$ for all $g \in G$. The restriction $\rho|_H : H \to \mathrm{GL}(V)$ is again a representation of H since H is a subgroup. Its character is the function

$$h \longmapsto \mathrm{tr}(\rho|_H(h)) = \chi(h) \qquad h \in H,$$

which is precisely χ_H. Therefore χ_H is a character of H. □

However, the restriction of an irreducible character of G to a subgroup H need not be irreducible. For example, let $G = S_3$ and $H = A_3$. Let χ be the irreducible character of S_3 of degree 2. Its values are

$$\chi(1) = 2, \qquad \chi(\text{transposition}) = 0, \qquad \chi(\text{3-cycle}) = -1.$$

Restricting χ to $H = \{1, (123), (132)\}$, we obtain

$$\chi_H(1) = 2, \qquad \chi_H((123)) = -1, \qquad \chi_H((132)) = -1.$$

Thus χ_H has degree 2. Since $A_3 \simeq C_3$ is abelian, all its irreducible complex characters have degree 1. Hence χ_H cannot be irreducible.

Indeed, if $\omega = e^{2\pi i/3}$ and φ is the character of A_3 defined by

$$\varphi(1) = 1, \qquad \varphi((123)) = \omega, \qquad \varphi((132)) = \omega^2,$$

then its complex conjugate character $\overline{\varphi}$ is given by

$$\overline{\varphi}(1) = 1, \qquad \overline{\varphi}((123)) = \omega^2, \qquad \overline{\varphi}((132)) = \omega.$$

Therefore

$$(\varphi + \overline{\varphi})(1) = 2,$$

while

$$(\varphi + \overline{\varphi})((123)) = \omega + \omega^2 = -1$$

and

$$(\varphi + \overline{\varphi})((132)) = \omega^2 + \omega = -1.$$

Hence $\chi_H = \varphi + \overline{\varphi}$.

Therefore the restriction of an irreducible character may be reducible.

Conversely, it is natural to ask how a class function defined on $H \leq G$ can be extended to a class function on G.

Definition 2.1.3 (Extension by zero)**.** Let $H \leq G$ and let $\varphi \in \mathrm{cf}(H)$. We define the function $\varphi^0 : G \to \mathbb{C}$ by

$$\varphi^0(g) = \begin{cases} \varphi(g), & \text{if } g \in H, \\ 0, & \text{if } g \notin H. \end{cases}$$

In general, φ^0 need not be a class function on G, because a conjugate of an element of H may lie outside H. By definition, if $H = G$, then $\varphi^0 = \varphi$.

Definition 2.1.4 (Induced class function)**.** Let $H \leq G$ and let $\varphi \in \mathrm{cf}(H)$. We define the *induced class function* $\varphi^G : G \to \mathbb{C}$ by

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^0(x^{-1}gx) \qquad \forall g \in G.$$

The function φ^G is a class function on G. Indeed, for $t \in G$ we have

$$\varphi^G(t^{-1}gt) = \frac{1}{|H|} \sum_{x \in G} \varphi^0(x^{-1}(t^{-1}gt)x) = \frac{1}{|H|} \sum_{y \in G} \varphi^0(y^{-1}gy) = \varphi^G(g),$$

where we used the change of variables $y = tx$.

Lemma 2.1.5. *Let $H \leq G$ and let $\varphi \in \mathrm{cf}(H)$. Then*

$$\varphi^G(1) = [G : H]\, \varphi(1).$$

Proof. Since $1^x = 1 \in H$ for all $x \in G$, we have

$$\varphi^G(1) = \frac{1}{|H|} \sum_{x \in G} \varphi^0(1^x) = \frac{1}{|H|} \sum_{x \in G} \varphi(1) = \frac{|G|}{|H|}\varphi(1) = [G : H]\varphi(1).$$

□

We now present a fundamental result, known as *Frobenius reciprocity*, which expresses the compatibility between induction and restriction; see [Isa94, Lemma 15.19].

Proposition 2.1.6 (Frobenius reciprocity)**.** *Let $H \leq G$, let $\varphi \in \mathrm{cf}(H)$ and let $\psi \in \mathrm{cf}(G)$. Then*

$$\langle \varphi^G, \psi \rangle_G = \langle \varphi, \psi_H \rangle_H$$

where $\langle \cdot, \cdot \rangle_G$ denotes the inner product on $\mathrm{cf}(G)$, while $\langle \cdot, \cdot \rangle_H$ denotes the inner product on $\mathrm{cf}(H)$.

Proof. By definition of the inner product on $\mathrm{cf}(G)$ and of the induced class function on G the following holds

$$\langle \varphi^G, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \varphi^G(g)\, \overline{\psi(g)} = \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{x \in G} \varphi^0(x^{-1}gx) \right) \overline{\psi(g)}.$$

Interchanging the sums gives

$$\langle \varphi^G, \psi \rangle_G = \frac{1}{|G||H|} \sum_{x \in G} \sum_{g \in G} \varphi^0(x^{-1}gx)\, \overline{\psi(g)}.$$

For each fixed $x \in G$, make the change of variables $y = x^{-1}gx$. Then $g = xyx^{-1}$ and, since ψ is a class function on G, we have $\psi(xyx^{-1}) = \psi(y)$. Hence

$$\sum_{g \in G} \varphi^0(x^{-1}gx)\, \overline{\psi(g)} = \sum_{y \in G} \varphi^0(y)\, \overline{\psi(y^{x^{-1}})} = \sum_{y \in G} \varphi^0(y)\, \overline{\psi(y)}.$$

Therefore we obtain

$$\langle \varphi^G, \psi \rangle_G = \frac{1}{|G||H|} \sum_{x \in G} \sum_{y \in G} \varphi^0(y)\, \overline{\psi(y)} = \frac{1}{|H|} \sum_{y \in G} \varphi^0(y)\, \overline{\psi(y)}.$$

Finally, since $\varphi^0(y) = 0$ for $y \in G \setminus H$, the sum reduces to the contribution given by the elements in H and

$$\langle \varphi^G, \psi \rangle_G = \frac{1}{|H|} \sum_{y \in H} \varphi(y)\, \overline{\psi(y)} = \langle \varphi, \psi_H \rangle_H,$$

as claimed. □

Corollary 2.1.7. *Let $H \leq G$ and let $\varphi \in \mathrm{Char}(H)$. Then the induced class function φ^G is a character of G, that is, $\varphi^G \in \mathrm{Char}(G)$.*

Proof. By Corollary 1.1.13, it suffices to show that

$$\langle \varphi^G, \chi \rangle_G \in \mathbb{Z}_{\geq 0} \qquad \text{for all } \chi \in \mathrm{Irr}(G).$$

Moreover, observe that $\varphi^G \neq 0$ since by Lemma 2.1.5 $\varphi^G(1) = [G : H]\ \varphi(1) \neq 0$.
Fix $\chi \in \mathrm{Irr}(G)$. By the Frobenius reciprocity Proposition 2.1.6, we know that

$$\langle \varphi^G, \chi \rangle_G = \langle \varphi, \chi_H \rangle_H.$$

Since $\chi \in \mathrm{Char}(G)$, Proposition 2.1.2 implies that $\chi_H \in \mathrm{Char}(H)$. Therefore both φ and χ_H are characters of H. By Corollary 1.1.13, we see that $\langle \varphi, \chi_H \rangle_H$ is a nonnegative integer. Hence $\langle \varphi^G, \chi \rangle_G \in \mathbb{Z}_{\geq 0}$ for all $\chi \in \mathrm{Irr}(G)$, and the result holds. □

2.2 Frobenius complements and key lemmas

As we have just seen, induction yields not only a class function: when we start from a character of H, the induced function is again a character of G. In general, however, restriction and induction are not inverse operations. Under a suitable hypothesis on the subgroup $H \leq G$, the two operations behave much more rigidly, as the next lemma shows. We first introduce the key concept of Frobenius complement.

Definition 2.2.1 (Frobenius complement)**.** Let G be a finite group and let $1 < H < G$ be a nontrivial proper subgroup. We say that H is a *Frobenius complement* in G if

$$H \cap H^g = 1 \qquad \text{for every } g \in G \setminus H,$$

where $H^g = g^{-1}Hg$.

Lemma 2.2.2. *Let H be a Frobenius complement in G, and $\varphi, \theta \in \mathrm{cf}(H)$ with $\varphi(1) = 0$. Then the following holds:*

1. $(\varphi^G)_H = \varphi$.
2. $\langle \theta^G, \varphi^G \rangle_G = \langle \theta, \varphi \rangle_H$.

Proof. We first establish 1. and then turn to 2.

Let $\varphi \in \mathrm{cf}(H)$ with $\varphi(1) = 0$. Recall that

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^0(x^{-1}gx) \qquad \forall g \in G,$$

where φ^0 is the extension by zero outside H. Let $h \in H$.

If $h = 1$, then by Lemma 2.1.5,

$$(\varphi^G)_H(1) = \varphi^G(1) = [G : H]\varphi(1) = 0 = \varphi(1).$$

Suppose now that $h \in H$ with $h \neq 1$. We compute $\varphi^G(h)$. Let $x \notin H$. Then $x^{-1} \notin H$, since H is a subgroup. Hence the Frobenius complement condition applies to x^{-1}, and so $H \cap H^{x^{-1}} = 1$. Since $h \in H$ and $h \neq 1$, it follows that $h \notin H^{x^{-1}} = xHx^{-1}$. Equivalently, $x^{-1}hx \notin H$, and therefore $\varphi^0(x^{-1}hx) = 0$, whenever $x \notin H$.

Thus, in the sum defining $\varphi^G(h)$, all terms with $x \notin H$ vanish. Hence

$$\varphi^G(h) = \frac{1}{|H|} \sum_{x \in G} \varphi^0(x^{-1}hx) = \frac{1}{|H|} \sum_{x \in H} \varphi^0(x^{-1}hx) = \frac{1}{|H|} \sum_{x \in H} \varphi(x^{-1}hx).$$

Since φ is a class function on H, we have $\varphi(x^{-1}hx) = \varphi(h)$ for all $x \in H$. Thus

$$\varphi^G(h) = \frac{1}{|H|} \sum_{x \in H} \varphi(h) = \varphi(h).$$

This completes the proof of 1.

We now turn to 2. Let $\varphi, \theta \in \mathrm{cf}(H)$ with $\varphi(1) = 0$. By Frobenius reciprocity Proposition 2.1.6, we have

$$\langle \theta^G, \varphi^G \rangle_G = \langle \theta, (\varphi^G)_H \rangle_H.$$

By item 1., we have $(\varphi^G)_H = \varphi$, hence

$$\langle \theta^G, \varphi^G \rangle_G = \langle \theta, \varphi \rangle_H,$$

as required. □

The behaviour described in Lemma 2.2.2 is a special case of the following more general result, known as the Brauer–Suzuki Theorem. A proof can be found in [Gor80, Theorem 4.6]. In this theorem, the subgroup H is the normalizer in G of a subset A which is disjoint from its conjugates outside H. Lemma 2.2.2 corresponds to the special case where $A = H \setminus \{1\}$, with H a Frobenius complement of G.

Theorem 2.2.3 (Brauer–Suzuki)**.** *Let G be a finite group and let $A \subseteq G$.*
Set $H = N_G(A) = \{\, g \in G \mid A^g = A \,\}$, where $A^g = g^{-1}Ag$. Assume that

$$A \cap A^g = \varnothing \qquad \text{for all } g \in G \setminus H.$$

Let $\chi, \theta \in \mathrm{cf}(H)$ be such that $\chi(h) = \theta(h) = 0$ for all $h \in H \setminus A$.

Then:

1. *$\chi^G(a) = \chi(a)$ for every nonidentity element $a \in A$.*
2. *if $\chi(1) = 0$, then $\langle \chi, \theta \rangle_H = \langle \chi^G, \theta^G \rangle_G$.*

Observe that the function φ appearing in Lemma 2.2.2 is not, in general, a character, because of the condition $\varphi(1) = 0$. Indeed, if χ is a nonzero character of a finite group, then $\chi(1)$ equals the degree of χ, and therefore

$$\chi(1) > 0.$$

For this reason, it may at first seem unclear how Lemma 2.2.2 can be applied in the study of characters. To overcome this difficulty, we introduce a notion that generalizes the concept of character.

A *generalized character* of a finite group G is a function $\varphi \in \mathrm{cf}(G)$ of the form

$$\varphi = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi, \qquad a_\chi \in \mathbb{Z}.$$

This notion is essential for the next lemma. Starting from an irreducible character of a subgroup H of a group G, we will construct a generalized character φ satisfying the hypothesis $\varphi(1) = 0$ of Lemma 2.2.2. This will allow us to prove not only the existence of an irreducible character of G, but also to obtain an explicit construction of it.

Lemma 2.2.4. *Let $H \leq G$ be a Frobenius complement and let $\chi \in \mathrm{Irr}(H)$. Then there exists $\chi^* \in \mathrm{Irr}(G)$ such that:*

1. $(\chi^*)_H = \chi$;
2. $\ker(\chi^*)$ *contains all elements of G that are not conjugate to any element of H.*

Proof. We will deal with the construction of the character χ^*. If $\chi = 1_H$, we simply take the character with constant value one, $\chi^* = 1_G$. Then $(\chi^*)_H = (1_G)_H = 1_H = \chi$, and condition 2. is also satisfied, since $\ker(1_G) = G$ contains every element of G.

Assume now that $\chi \neq 1_H$, so χ is a nontrivial irreducible character of H. Set

$$\varphi = \chi - \chi(1)\, 1_H \in \mathrm{cf}(H).$$

Then φ is a generalized character of H (i.e. an integer linear combination of characters). Moreover, $\varphi(1) = \chi(1) - \chi(1) = 0$, so Lemma 2.2.2 applies.

Consider the induced function φ^G. Since φ is a difference of characters of H, and since, by Corollary 2.1.7, the induced function of a character of H is a character of G, it follows that φ^G is a generalized character of G.

We now compute the coefficient of the trivial character 1_G in φ^G. By Frobenius reciprocity Proposition 2.1.6,

$$\langle \varphi^G, 1_G \rangle_G = \langle \varphi, (1_G)_H \rangle_H = \langle \varphi, 1_H \rangle_H = \langle \chi, 1_H \rangle_H - \chi(1) \langle 1_H, 1_H \rangle_H.$$

Since $\chi \in \mathrm{Irr}(H)$ is nontrivial, we have $\langle \chi, 1_H \rangle_H = 0$, and of course $\langle 1_H, 1_H \rangle_H = 1$. Hence

$$\langle \varphi, 1_H \rangle_H = \langle \chi - \chi(1)1_H,\ 1_H \rangle_H = 0 - \chi(1) \cdot 1 = -\chi(1).$$

Therefore $\langle \varphi^G, 1_G \rangle_G = -\chi(1)$, and we may write

$$\varphi^G = \chi^* - \chi(1)\, 1_G,$$

where χ^* is a generalized character of G with no trivial constituent, equivalently $\langle \chi^*, 1_G \rangle_G = 0$. Therefore

$$\chi^* = \varphi^G + \chi(1)\, 1_G.$$

Finally, by Lemma 2.2.2(1) we have $(\varphi^G)_H = \varphi$, and hence

$$(\chi^*)_H = (\varphi^G)_H + \chi(1)(1_G)_H = \varphi + \chi(1)\, 1_H = \chi.$$

To conclude the proof of 1. it remains to show that χ^* is an irreducible character of G.

By Lemma 2.2.2(2), we have

$$\langle \varphi^G, \varphi^G \rangle_G = \langle \varphi, \varphi \rangle_H = \langle \chi - \chi(1)1_H,\ \chi - \chi(1)1_H \rangle_H = \langle \chi,\ \chi \rangle_H + \chi(1)^2 = 1 + \chi(1)^2.$$

Hence

$$\langle \varphi^G, \varphi^G \rangle_G = 1 + \chi(1)^2. \tag{2.2.1}$$

On the other hand,

$$\langle \varphi^G, \varphi^G \rangle_G = \langle \chi^* - \chi(1)1_G,\ \chi^* - \chi(1)1_G \rangle_G = \langle \chi^*, \chi^* \rangle_G + \chi(1)^2. \tag{2.2.2}$$

Combining (2.2.1) and (2.2.2), we conclude that

$$\langle \chi^*, \chi^* \rangle_G = 1.$$

Since χ^* is a generalized character of G, we can write

$$\chi^* = \sum_{\psi \in \mathrm{Irr}(G)} a_\psi\, \psi \qquad \text{with} \qquad a_\psi \in \mathbb{Z}.$$

Using the orthonormality of $\mathrm{Irr}(G)$ Theorem 1.1.10 we obtain

$$1 = \langle \chi^*, \chi^* \rangle_G = \sum_{\psi \in \mathrm{Irr}(G)} a_\psi^2.$$

It follows that exactly one coefficient is nonzero and it must be ± 1. Hence $\chi^* = \pm\psi$ for some $\psi \in \mathrm{Irr}(G)$. Finally,

$$\chi^*(1) = \varphi^G(1) + \chi(1) = [G : H]\varphi(1) + \chi(1) = \chi(1) > 0,$$

and we cannot have $\chi^* = -\psi$, but necessarily we have $\chi^* = \psi \in \mathrm{Irr}(G)$. This completes the proof of 1.

We now prove assertion 2.. Let $g \in G$ be an element that is not conjugate to any element of H. We claim that $g \in \ker(\chi^*)$. Indeed, we recall that

$$\varphi^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^0(x^{-1}gx),$$

and, since g is not conjugate to any element of H, we have $x^{-1}gx \notin H$ for any $x \in G$. Therefore $\varphi^0(x^{-1}gx) = 0$ for all $x \in G$, and this yields that

$$\varphi^G(g) = 0.$$

By definition, $\chi^* = \varphi^G + \chi(1)1_G$. Thus, if $g \in G$ is an element that is not conjugate to any element of H, then

$$\chi^*(g) = \varphi^G(g) + \chi(1)1_G(g) = 0 + \chi(1) = \chi(1).$$

Moreover,

$$\chi^*(1) = \varphi^G(1) + \chi(1) = [G : H]\varphi(1) + \chi(1) = \chi(1),$$

since $\varphi(1) = 0$. Hence $\chi^*(g) = \chi^*(1)$, and so $g \in \ker(\chi^*)$ as claimed.

□

2.3 The proof of Frobenius' Theorem and examples

We are now ready to prove the main result of this chapter, Frobenius' Theorem [Isa94, Theorem 15.1].

Theorem 2.3.1 (Frobenius)**.** *Let G be a finite group and let $H < G$ be a Frobenius complement. Then the subset*

$$N = \{\, g \in G \mid g \text{ is not conjugate to any nonidentity element of } H \}$$

is a normal subgroup of G, $G = NH$, and $N \cap H = 1$.

Proof. We recall that $H \cap H^g = 1 \quad \forall g \in G \setminus H$. Let us consider the set

$$N = \Big(G \setminus \bigcup_{g \in G} H^g\Big) \cup \{1\},$$

and show that N is a normal subgroup of G.

Let $\chi \in \mathrm{Irr}(H)$. By Lemma 2.2.4, there exists $\chi^* \in \mathrm{Irr}(G)$ such that $\ker(\chi^*)$ contains every element of G that is not conjugate to any element of H. In particular, every element of $G \setminus \bigcup_{g \in G} H^g$ is not conjugate to any element of H, hence

$$N \subseteq \ker(\chi^*) \qquad \text{for every } \chi \in \mathrm{Irr}(H). \tag{2.3.1}$$

Let

$$M = \bigcap_{\chi \in \mathrm{Irr}(H)} \ker(\chi^*).$$

Each $\ker(\chi^*)$ is normal in G, so M is normal in G. Moreover, from (2.3.1) we have $N \subseteq M$.

We claim that $M \cap H = 1$. Indeed, if $h \in M \cap H$, then $h \in \ker(\chi^*)$ for every $\chi \in \mathrm{Irr}(H)$, so $\chi^*(h) = \chi^*(1)$ for all $\chi \in \mathrm{Irr}(H)$. Restricting to H and using the fact that $\chi^*_H = \chi$ by Lemma 2.2.4(1), we obtain $\chi(h) = \chi(1)$ for all $\chi \in \mathrm{Irr}(H)$. Hence $h \in \bigcap_{\chi \in \mathrm{Irr}(H)} \ker(\chi)$, whenever $h \in M \cap H$. It follows from Corollary 1.1.19, applied to H, that

$$M \cap H = \bigcap_{\chi \in \mathrm{Irr}(H)} \ker(\chi) = 1.$$

Since $M \trianglelefteq G$, we have $M \cap H^g = (M^{g^{-1}} \cap H)^g = (M \cap H)^g = 1$, for every $g \in G$. Therefore no element of $M \setminus \{1\}$ lies in any conjugate H^g, which means

$$M \subseteq \big(G \setminus \bigcup_{g \in G} H^g\big) \cup \{1\} = N.$$

We conclude that $M = N$. In particular, $N \lhd G$ and $N \cap H = 1$.

It remains to prove that $G = NH$. Since we already know that $N \lhd G$ and $N \cap H = 1$, it is enough to compute $|N|$ and show that $|N| = |G|/|H|$. First observe that

$$N_G(H) = H. \tag{2.3.2}$$

Indeed, if $g \in N_G(H)$, then $H^g = H$, and hence $H \cap H^g = H \neq 1$. So by the Frobenius complement condition we must have $g \in H$. Now consider the action of G by conjugation on the set of subgroups of G. Under this action, the orbit of H is $H^G = \{gHg^{-1} \mid g \in G\}$.

Moreover, the stabilizer of H is precisely its normalizer:

$$\mathrm{Stab}_G(H) = \{g \in G \mid H^g = H\} = N_G(H).$$

Hence, by the orbit-stabilizer Theorem 1.2.1 and (2.3.2), we have

$$|H^G| = [G : \mathrm{Stab}_G(H)] = [G : N_G(H)] = [G : H].$$

Therefore the number of distinct conjugates of H in G is $[G : H]$.

Moreover, if $H^g \neq H^k$, then their intersection is trivial. Indeed,

$$H^g \cap H^k = \big(H \cap H^{kg^{-1}}\big)^g,$$

and $H^{kg^{-1}} \neq H$ implies $kg^{-1} \notin H$, hence $H \cap H^{kg^{-1}} = 1$ by the Frobenius complement condition. Consequently,

$$(H^g \setminus \{1\}) \cap (H^k \setminus \{1\}) = \varnothing \qquad \text{whenever } H^g \neq H^k.$$

Thus the union of the sets $H^g \setminus \{1\}$ is a disjoint union over the $[G : H]$ distinct conjugates. We obtain

$$\Big|\bigcup_{g\in G}\big(H^g \setminus \{1\}\big)\Big| = [G : H]\,(|H| - 1),$$

and so

$$N = \Big(G \setminus \bigcup_{g\in G} H^g\Big) \cup \{1\} = G \setminus \bigcup_{g\in G}\big(H^g \setminus \{1\}\big),$$

so

$$|N| = |G| - \Big|\bigcup_{g\in G}\big(H^g \setminus \{1\}\big)\Big| = |G| - [G : H](|H| - 1) = [G : H].$$

Finally, since $N \cap H = 1$, we have $|NH| = |N||H| = |G|$, and hence $NH = G$. □

Theorem 2.3.1 isolates a class of finite groups in which the existence of a suitable complement forces the presence of a distinguished normal subgroup.

Definition 2.3.2 (Frobenius group)**.** A finite group G is called a *Frobenius group* if it admits a Frobenius complement H with

$$1 < H < G.$$

In this case, the normal subgroup N provided by Theorem 2.3.1 is called the *Frobenius kernel* of G. In particular, the Frobenius group G is the semidirect product of its Frobenius

kernel N and a Frobenius complement H:

$$G = N \rtimes H.$$

Theorem 2.3.1 also gives a natural decomposition of the elements of a Frobenius group. Recall that a partition of a finite group G is a collection Π of nontrivial proper subgroups of G such that

$$G = \bigcup_{X \in \Pi} X$$

and

$$X \cap Y = 1 \quad \text{for all distinct } X, Y \in \Pi.$$

Definition 2.3.3 (Frobenius partition)**.** Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. The collection

$$\Pi = \{N\} \cup \{H^g \mid g \in G\}$$

is called the *Frobenius partition* of G. Equivalently, the elements of G decompose as

$$G = N \cup \bigcup_{g \in G} \big(H^g \setminus \{1\}\big).$$

The union in Definition 2.3.3 is disjoint: if $H^g \neq H^k$, then $H^g \cap H^k = 1$, hence $(H^g \setminus \{1\}) \cap (H^k \setminus \{1\}) = \varnothing$. This is the same counting argument used in the proof of Theorem 2.3.1.

Frobenius partitions provide an important tool for studying the structure of Frobenius groups. They describe how the elements of the group are distributed between the kernel and the conjugates of a complement.

A natural question is whether this partition is uniquely determined by the group, or whether different Frobenius partitions may exist. Answering this question requires further structural properties of Frobenius groups and will be addressed later.

We conclude this chapter with concrete examples illustrating Frobenius' Theorem.

Example 2.3.4 (The group S_3)**.** Consider the symmetric group $G = S_3$, the group of all permutations of the set $\{1, 2, 3\}$. Recall that $|S_3| = 6$.
Let $H = \langle (1\,2) \rangle = \{1, (1\,2)\}$. Then H is a subgroup of order 2.

We now compute the conjugates of H in S_3. Since conjugation preserves cycle type, the conjugates of the transposition $(1\,2)$ are precisely the transpositions of S_3. Hence the conjugates of H are

$$\langle (1\,2) \rangle, \qquad \langle (1\,3) \rangle, \qquad \langle (2\,3) \rangle.$$

Any two distinct conjugates of H intersect trivially. By Frobenius' Theorem 2.3.1, G admits a normal subgroup N such that $G = NH, N \cap H = 1$. In this example,

$$N = \langle (1\,2\,3) \rangle = \{1, (1\,2\,3), (1\,3\,2)\}.$$

Indeed, N is the unique subgroup of order 3 of S_3, hence it is normal. Moreover, $|N||H| = 3 \cdot 2 = 6 = |S_3|$, and since $N \cap H = 1$, it follows that

$$S_3 = N \rtimes H.$$

Thus S_3 is a Frobenius group with Frobenius kernel

$$N = \langle (1\,2\,3) \rangle \cong C_3$$

and Frobenius complement

$$H = \langle (1\,2) \rangle \cong C_2.$$

This is the smallest nontrivial example of a Frobenius group. It also provides a first illustration of the semidirect product decomposition described by Frobenius' Theorem.

Example 2.3.5 (Groups of order pq)**.** The previous example can be generalized in a natural way. We now construct a family of Frobenius groups of order pq, where p and q are distinct prime numbers satisfying $p \mid (q-1)$. Recall that the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$ has order $q-1$. Since $p \mid (q-1)$, Cauchy's Theorem 1.2.14 implies that there exists an element $r \in (\mathbb{Z}/q\mathbb{Z})^\times$ of order p. Equivalently, $r^p \equiv 1 \pmod{q}$, while

$$r^m \not\equiv 1 \pmod{q} \qquad \text{for all } 0 < m < p.$$

Consider the non-abelian group of order pq, where $p|(q-1)$

$$G = \langle a, b \mid a^q = b^p = 1,\ b^{-1}ab = a^r \rangle.$$

The subgroup $N = \langle a \rangle$ has order q, while $H = \langle b \rangle$ has order p. Since the relation

$$b^{-1}ab = a^r$$

shows that conjugation by b sends powers of a to powers of a, the subgroup N is normal in G. Thus $G = N \rtimes H$.

We now verify that H is a Frobenius complement for G.

Let $g \in G \setminus H$. Since $G = NH = HN$, every element of G can be written uniquely

in the form

$$g = b^t a^u, \qquad 0 \le t < p, \quad 0 \le u < q.$$

Since $g \notin H$, we have $u \neq 0$.

Let $h = b^m \in H$ be nontrivial, so $0 < m < p$. We compute the conjugate of h by g:

$$g^{-1}hg = (b^t a^u)^{-1} b^m (b^t a^u) = a^{-u} b^{-t} b^m b^t a^u = a^{-u} b^m a^u. \tag{2.3.3}$$

From the relation $b^{-1}ab = a^r$ it follows that $b^{-1}a^v b = a^{vr}$ for every $v \in \mathbb{Z}$. Equivalently,

$$a^v b = b a^{vr}.$$

Iterating this identity, we get

$$a^v b^m = b^m a^{vr^m} \qquad \text{for every } v \in \mathbb{Z}.$$

Taking $v = -u$, we obtain

$$a^{-u} b^m = b^m a^{-ur^m}.$$

Therefore (2.3.3) becomes

$$g^{-1}hg = a^{-u} b^m a^u = b^m a^{-ur^m} a^u = b^m a^{u(1-r^m)}.$$

Now $0 < m < p$, and r has order p modulo q. Therefore $r^m \not\equiv 1 \pmod{q}$, so $1 - r^m \not\equiv 0 \pmod{q}$. Since also $u \neq 0$, we conclude that $u(1 - r^m) \not\equiv 0 \pmod{q}$. Thus $a^{u(1-r^m)} \neq 1$, and consequently $g^{-1}hg \notin H$.

We have shown that no nontrivial element of H belongs to $H \cap H^g$ when $g \notin H$. Therefore

$$H \cap H^g = 1 \qquad \text{for every } g \in G \setminus H.$$

Hence H is a Frobenius complement in G, and G is a Frobenius group with Frobenius kernel

$$N = \langle a \rangle \cong C_q$$

and Frobenius complement

$$H = \langle b \rangle \cong C_p.$$

The example of S_3 considered above corresponds to the special case $q = 3$ and $p = 2$. Indeed, $S_3 \cong C_3 \rtimes C_2$. This family provides one of the simplest and most important classes of Frobenius groups. In particular, it shows how Frobenius groups naturally arise as semidirect products.

Chapter 3

Frobenius groups

The purpose of this chapter is to study Frobenius groups from a more structural point of view, and to present the equivalent formulations that occur in the literature. Besides the definition given in Chapter 2 in terms of a Frobenius complement, there are two further viewpoints that are especially useful in the study of Frobenius groups. Mainly, one can consider:

- an automorphism description, where the Frobenius complement acts on the Frobenius kernel by conjugation through fixed-point-free automorphisms;
- a description in terms of group actions on finite sets, where Frobenius groups arise from suitable transitive actions.

We will show how these viewpoints are related, and how each of them highlights different aspects of the same structure.

3.1 Fixed-point-free automorphisms and centralizers

We first study Frobenius groups through the action of the complement on the kernel. This point of view follows directly from Frobenius' Theorem. Indeed, if G is a Frobenius group with Frobenius kernel N and complement H, then by Theorem 2.3.1 $N \trianglelefteq G$, and conjugation defines an action of H on N by automorphisms. The material in this section mainly follows [Isa08, Section 6A].

More precisely, the next theorem shows that the conjugation action of H on N is fixed-point-free and that Frobenius groups can be characterized in terms of such actions; see [Hup25, Theorem 8.5].

Theorem 3.1.1. *The following conditions are equivalent:*

1. *G is a Frobenius group with Frobenius complement H and Frobenius kernel N.*

2. *We have $G = NH$ for some $N, H \leq G$, with $N \lhd G$ and $1 < H < G$ and the conjugation map*

$$\varphi : H \longrightarrow \mathrm{Aut}(N), \qquad \varphi(h)(n) = hnh^{-1},$$

is an isomorphism from H onto a fixed-point-free subgroup of $\mathrm{Aut}(N)$.

Proof. Assume 1., that is, that G is a Frobenius group with Frobenius complement H and Frobenius kernel N. By Frobenius' Theorem 2.3.1, we have $G = NH$, with $N \lhd G$ and $N \cap H = 1$. We prove that the conjugation action of H on N is fixed-point-free. Equivalently, for every $h \in H \setminus \{1\}$,

$$\mathrm{Fix}_N(h) = \{\, n \in N \mid hnh^{-1} = n \,\} = \{1\}.$$

Let $h \in H \setminus \{1\}$ and suppose that $n \in N$ satisfies $hnh^{-1} = n$. Then $h = n^{-1}hn$, so $h \in H^n$. Hence $h \in H \cap H^n$.

If $n \neq 1$, then $n \notin H$ because $N \cap H = 1$, and the Frobenius complement condition gives $H \cap H^n = 1$, contradicting $h \neq 1$. Therefore $n = 1$, and so $\mathrm{Fix}_N(h) = \{1\}$. Thus every nonidentity element of H induces a fixed-point-free automorphism of N.

Therefore φ is an isomorphism from H onto the fixed-point-free subgroup $\varphi(H) \leq \mathrm{Aut}(N)$. Thus condition 2. is proved.

Conversely, assume that $G = NH$, with $N, H \leq G$, with $N \lhd G$ and $1 < H < G$, and the conjugation action of H on N is fixed-point-free.

We first show that $N \cap H = 1$. Let $x \in N \cap H$. Since $x \in H$, it acts on N by conjugation, and since $x \in N$, we have $xxx^{-1} = x$. Hence x is a fixed point of the automorphism induced by x. If $x \neq 1$, this contradicts the fixed-point-free assumption. Therefore $x = 1$, and so $N \cap H = 1$.

We now prove that H is a Frobenius complement in G, namely that $H \cap H^g = 1$ for all $g \in G \setminus H$.

Let $g \in G \setminus H$. Since $G = NH$, we can write $g = nh$ for some $n \in N$ and $h \in H$. Because $g \notin H$ and $N \cap H = 1$, necessarily $n \neq 1$. We may rewrite

$$g = nh = hh^{-1}nh = h\,(h^{-1}nh).$$

Setting $f = h^{-1}nh$ which belongs to N because $N \lhd G$, we obtain $g = hf$. Finally, $f \neq 1$ because $f = 1$ would imply $n = hfh^{-1} = 1$, contradicting $n \neq 1$.

We claim that $H^g = H^f$. Indeed, using $g = hf$ we have

$$H^g = H^{hf} = (H^h)^f.$$

But $h \in H$, hence $H^h = H$, so $H^g = H^f$.

Now let $h_1 \in H \cap H^g$. Since $H^g = H^f$, we have $h_1 \in H \cap H^f$, and therefore there exists $h_2 \in H$ such that

$$h_1 = h_2^f = f^{-1}h_2 f. \tag{3.1.1}$$

By (3.1.1) we obtain

$$h_1 h_2^{-1} = (f^{-1}h_2 f)\, h_2^{-1}. \tag{3.1.2}$$

Then

$$[f, h_2^{-1}] = f^{-1}(h_2^{-1})^{-1} f h_2^{-1} = f^{-1}h_2 f h_2^{-1},$$

and combining with (3.1.2) we get

$$h_1 h_2^{-1} = f^{-1}h_2 f h_2^{-1} = [f, h_2^{-1}]. \tag{3.1.3}$$

Since $h_1, h_2 \in H$, the left-hand side of (3.1.3) lies in H. On the other hand, $f \in N$ and $N \lhd G$, so $f^{h_2^{-1}} \in N$, and therefore the right-hand side of (3.1.3) $[f, h_2^{-1}] \in N$. Thus

$$h_1 h_2^{-1} \in H \cap N.$$

But $H \cap N = 1$, so $h_1 h_2^{-1} = 1$ and hence $h_1 = h_2$. In combination with (3.1.1) we have

$$h_1 = f^{-1}h_1 f,$$

which is equivalent to $f h_1 = h_1 f$, and therefore to $f^{h_1} = f$. Since $f \neq 1$ and the action of H on N is fixed-point-free, this forces h_1 to be 1.

Thus we have shown that $H \cap N = 1$ and $H \cap H^g = 1$ for all $g \notin H$. Therefore G is a Frobenius group with complement H and kernel N. □

As an important consequence of Theorem 3.1.1, we translate the fixed-point-free action associated with a Frobenius group into a property of centralizers.

Corollary 3.1.2. *Let G be a Frobenius group with Frobenius kernel N and complement H. Then $C_N(h) = 1$ for every $h \in H \setminus \{1\}$.*

Proof. By Theorem 3.1.1, the action by conjugation of H on N is fixed-point-free. Hence

$$\mathrm{Fix}_N(h) = \{1\} \qquad \text{for every } h \in H \setminus \{1\}.$$

By Remark 1.2.6, we have $\mathrm{Fix}_N(h) = C_N(h)$, and therefore $C_N(h) = 1$ for every $h \in H \setminus \{1\}$. □

Theorem 3.1.1 and Corollary 3.1.2 show that the action of the complement on the kernel can be described equivalently in terms of fixed points or in terms of centralizers. This motivates the following terminology.

Definition 3.1.3 (Frobenius action by automorphisms)**.** Let H and N be finite groups, and suppose that H acts on N by automorphisms. We say that the action of H on N is a Frobenius action if

$$C_N(h) = 1$$

for every $h \in H \setminus \{1\}$.
Equivalently, no nonidentity element of H fixes a nonidentity element of N.

Corollary 3.1.2 shows that no non-trivial element of H centralizes a non-trivial element of N. The next proposition strengthens this observation by extending it to the whole group: we show that no element outside N, equivalently no element contained in a non-trivial conjugate of H, can centralize a non-trivial element of N. This is a standard property of Frobenius groups; cf. [Gor80, 7.6(4)].

Proposition 3.1.4. *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. Then for every $n \in N \setminus \{1\}$ one has*

$$C_G(n) \leq N.$$

Proof. Let $n \in N \setminus \{1\}$ and let $x \in C_G(n)$. We prove that $x \in N$.

Suppose, by contradiction, that $x \notin N$. By the Frobenius partition given in Definition 2.3.3,

$$G = N \cup \bigcup_{g \in G} \big(H^g \setminus \{1\}\big),$$

there exist $g \in G$ and $h \in H \setminus \{1\}$ such that $x = h^g = g^{-1}hg$.
Since $x \in C_G(n)$, we have

$$x^{-1}nx = n.$$

Substituting $x = g^{-1}hg$, this becomes $(g^{-1}hg)^{-1}n(g^{-1}hg) = n$. Equivalently,

$$g^{-1}h^{-1}gng^{-1}hg = n.$$

Multiplying on the left by g and on the right by g^{-1}, we obtain

$$h^{-1}gng^{-1}h = gng^{-1}. \tag{3.1.4}$$

Since $N \triangleleft G$ and $n \in N$, we have $gng^{-1} \in N$. Moreover, since $n \neq 1$, also $gng^{-1} \neq 1$.

Thus the equality (3.1.4) tells us that the non-trivial element $h^{-1} \in H$ fixes the non-trivial element $gng^{-1} \in N$ under the conjugation action of H on N. This contradicts Theorem 3.1.1, which says that the conjugation action of H on N is fixed-point-free. Therefore our assumption $x \notin N$ is false. Hence $x \in N$. Since every element of $C_G(n)$ lies in N, the result holds. □

We now present a general consequence of fixed-point-free actions by automorphisms, expressed through the equivalent centralizer condition. The result shows that these actions are necessarily coprime.

Lemma 3.1.5. *Let H be a finite group acting on a finite group N by automorphisms, and assume that $C_N(h) = 1$ for every $h \in H \setminus \{1\}$. Then*

$$|N| \equiv 1 \pmod{|H|}.$$

In particular, $|H|$ and $|N|$ are coprime.

Proof. Consider the action of H on the set N. The identity element of N is fixed by every element of H, so $\{1\}$ is an orbit of this action.

Let $n \in N \setminus \{1\}$. We claim that the stabilizer of n in H is trivial. Indeed, if $h \in \mathrm{Stab}_H(n)$, then $h \cdot n = n$. Thus n is a fixed point of h. If $h \neq 1$, this contradicts the hypothesis $C_N(h) = 1$. Therefore $h = 1$, and hence $\mathrm{Stab}_H(n) = 1$. Therefore, by the Orbit-stabilizer Theorem 1.2.1, the orbit of every element $n \in N \setminus \{1\}$ has size

$$|H : \mathrm{Stab}_H(n)| = |H|.$$

Thus $N \setminus \{1\}$ is a disjoint union of orbits, all of size $|H|$. Hence $|H| \mid (|N| - 1)$, that is $|N| \equiv 1 \pmod{|H|}$. In particular, if a prime divides both $|H|$ and $|N|$, then it also divides $|N| - 1$, and therefore it divides 1. Hence $(|H|, |N|) = 1$. □

As an immediate consequence of Corollary 3.1.2 and Lemma 3.1.5, we obtain the following divisibility property.

Corollary 3.1.6. *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. Then*

$$|H| \mid (|N| - 1).$$

In particular, H and N are Hall subgroups of G.

Proof. By Corollary 3.1.2, the action of H on N by conjugation satisfies $C_N(h) = 1$ for every $h \in H \setminus \{1\}$. Hence Lemma 3.1.5 gives $|N| \equiv 1 \pmod{|H|}$. Thus $|H| \mid (|N| - 1)$, and consequently $(|H|, |N|) = 1$. Since $|G| = |N||H|$, both N and H are Hall subgroups of G. □

We now record two consequences of the same centralizer condition. They show that this property is preserved when passing to subgroups of H and to suitable quotients of N.

Corollary 3.1.7. *Let H and N be finite groups, and suppose that H acts on N by automorphisms. Assume that the action is Frobenius, that is $C_N(h) = 1$ for every $h \in H \setminus \{1\}$. If $B \leq H$, then the induced action of B on N is also Frobenius.*

Proof. Let $b \in B \setminus \{1\}$. Since $B \leq H$, we also have $b \in H \setminus \{1\}$. Therefore, by hypothesis, $C_N(b) = 1$. Hence the action of B on N is Frobenius. □

Corollary 3.1.8. *Let H and N be finite groups, and suppose that H acts on N by automorphisms. Assume that this action is Frobenius. Let M be an H-invariant normal subgroup of N. Then the induced action of H on N/M is Frobenius, that is,*

$$C_{N/M}(h) = 1 \qquad \text{for every } h \in H \setminus \{1\}.$$

Proof. Since M is H-invariant, the action of H on N induces an action on the quotient N/M. We have to show that, for every $h \in H \setminus \{1\}$,

$$C_{N/M}(h) = 1.$$

Let $h \in H \setminus \{1\}$. Since $C_N(h) = 1$ for every $h \in H \setminus \{1\}$, by Lemma 3.1.5 the orders of H and N are coprime; in particular, $|\langle h \rangle|$ is coprime to $|M|$. Writing $\overline{N} = N/M$, and applying Corollary 1.2.9 to the action of $\langle h \rangle$ on N, we obtain

$$C_{\overline{N}}(\langle h \rangle) = C_N(\langle h \rangle)M/M.$$

Since $C_{\overline{N}}(\langle h \rangle) = C_{\overline{N}}(h)$ and $C_N(\langle h \rangle) = C_N(h)$, it follows that $C_{\overline{N}}(h) = C_N(h)M/M$. By hypothesis $C_N(h) = 1$, and therefore

$$C_{\overline{N}}(h) = M/M = 1.$$

Thus the induced action on N/M is a Frobenius action by automorphisms as desired. □

3.2 Frobenius actions and Frobenius permutation groups

We now turn to the point of view of group actions on finite sets. While the previous section described Frobenius groups through the action of the complement on the kernel by automorphisms, we now show that the same class of groups can also be characterized by a special kind of transitive action on a finite set.

We first introduce the notions of semiregular and regular actions.

Definition 3.2.1 (Semiregular and regular actions)**.** Let G act on a finite set Ω. The action is said to be *semiregular* if every nonidentity element fixes no point, i.e.

$$\mathrm{Fix}_\Omega(g) = \varnothing \qquad \text{for all } g \in G \setminus \{1\}.$$

The action is *regular* if it is both transitive and semiregular.

If the action of G on Ω is regular, then for every $\alpha, \beta \in \Omega$ there exists a unique element $g \in G$ such that $g \cdot \alpha = \beta$. Indeed, transitivity gives the existence of such an element. If $g_1, g_2 \in G$ both satisfy $g_1 \cdot \alpha = \beta = g_2 \cdot \alpha$, then $(g_2^{-1} g_1) \cdot \alpha = \alpha$. Since the action is semiregular, this implies $g_2^{-1} g_1 = 1$, hence $g_1 = g_2$.

We now isolate the action-theoretic condition that characterizes Frobenius groups; see [Ros94, Exercise 248].

Definition 3.2.2 (Frobenius action)**.** Let G act on a finite set Ω. The action is said to be a *Frobenius action* if it is transitive but not regular, $|\Omega| > 1$, and, for every two distinct elements $\alpha, \beta \in \Omega$, one has

$$\mathrm{Stab}_G(\alpha) \cap \mathrm{Stab}_G(\beta) = 1.$$

Equivalently, the action is transitive but not regular, $|\Omega| > 1$, and every nonidentity element of G fixes at most one point of Ω, that is,

$$g \neq 1 \implies |\mathrm{Fix}_\Omega(g)| \leq 1.$$

Indeed, an element fixes two distinct points α and β if and only if it belongs to the intersection $\mathrm{Stab}_G(\alpha) \cap \mathrm{Stab}_G(\beta)$.

We now show that this formulation is equivalent to the definition of Frobenius groups given in Chapter 2; this equivalence is stated in [Hup25, Theorem 8.2].

Theorem 3.2.3. 1. *Let $G \leq \mathrm{Sym}(\Omega)$ act on a finite set Ω, and assume that the action is a Frobenius action. Fix $\alpha \in \Omega$ and set $H = \mathrm{Stab}_G(\alpha)$. Then G is a Frobenius group with Frobenius complement H. Moreover, if*

$$F = \{\, g \in G \mid \mathrm{Fix}_\Omega(g) = \varnothing \,\} \cup \{1\},$$

then F coincides with the Frobenius kernel of G, and F acts regularly on Ω.

2. *Conversely, let G be a Frobenius group with Frobenius complement H and Frobenius kernel N. Consider the action of G on the set Ω of left cosets of H in G by left multiplication. Then the action of G on Ω is a Frobenius action.*

Proof. 1. Fix $\alpha \in \Omega$ and set $H = \mathrm{Stab}_G(\alpha)$. We first prove that

$$H \cap H^g = 1 \qquad \text{for all } g \in G \setminus H.$$

Let $g \in G \setminus H$. Then $g^{-1} \notin H$, and hence g^{-1} does not fix α. Thus $\beta = g^{-1} \cdot \alpha$ is different from α.

Let $x \in H \cap H^g$. Since $x \in H = \mathrm{Stab}_G(\alpha)$, we have $x \cdot \alpha = \alpha$. On the other hand, $x \in H^g$ means that $gxg^{-1} \in H = \mathrm{Stab}_G(\alpha)$, hence

$$(gxg^{-1}) \cdot \alpha = \alpha. \tag{3.2.1}$$

Applying g^{-1} to both sides of (3.2.1) and using the action property, this becomes

$$x \cdot (g^{-1} \cdot \alpha) = g^{-1} \cdot \alpha,$$

that is $x \cdot \beta = \beta$. Therefore $x \in \mathrm{Stab}_G(\alpha) \cap \mathrm{Stab}_G(\beta)$. Since $\alpha \neq \beta$, the definition of Frobenius action gives $x = 1$. Thus $H \cap H^g = 1$. Since the action is transitive and $|\Omega| > 1$, the stabilizer H is proper in G. Moreover, since the action is not regular, $H \neq 1$. Therefore $1 < H < G$, and so G is a Frobenius group with Frobenius complement H.

Let N be the Frobenius kernel given by Frobenius' Theorem 2.3.1, so that

$$N = \Big(G \setminus \bigcup_{x \in G} H^x\Big) \cup \{1\}.$$

We claim that $F = N$, where $F = \{\, g \in G \mid \mathrm{Fix}_\Omega(g) = \varnothing \,\} \cup \{1\}$.

To prove the claim, we first show that an element $g \in G$ fixes some point of Ω if and only if it belongs to a conjugate of H. Since the action of G is transitive, every point of Ω is of the form $x \cdot \alpha$ for some $x \in G$. Now g fixes $x \cdot \alpha$ if and only if

$$g \cdot (x \cdot \alpha) = x \cdot \alpha \iff (x^{-1}gx) \cdot \alpha = \alpha \iff x^{-1}gx \in H \iff g \in H^{x^{-1}}.$$

Since x^{-1} ranges over G as x ranges over G, we get

$$\{\, g \in G \mid \mathrm{Fix}_\Omega(g) \neq \varnothing \,\} = \bigcup_{x \in G} H^x.$$

Taking complements in G and then adding $\{1\}$ yields

$$F = \Big(G \setminus \bigcup_{x \in G} H^x\Big) \cup \{1\} = N,$$

as claimed. In particular, by Theorem 2.3.1, F is normal in G.

Finally, we show that F acts regularly on Ω. By definition of F, every nonidentity element of F fixes no point of Ω, so the action of F on Ω is semiregular. Moreover, by Frobenius' Theorem 2.3.1 we have $G = FH$ and $F \cap H = 1$. Therefore

$$[F : \mathrm{Stab}_F(\alpha)] = [F : F \cap H] = |F| = [G : H] = |\Omega|$$

because the action of G is transitive, and so F acts transitively on Ω as well. Thus F is both transitive and semiregular, i.e. it acts regularly on Ω, as desired.

2. Let G be a Frobenius group with complement H and kernel N, so $1 < H < G$ and

$$H \cap H^g = 1 \qquad \text{for all } g \in G \setminus H.$$

Consider the action of G on $\Omega = \{gH \mid g \in G\}$ by left multiplication. This action is transitive because for any two left cosets xH and yH we have $(yx^{-1}) \cdot (xH) = yH$. Moreover, $|\Omega| = [G : H] > 1$, since $H < G$. The action is not regular because the stabilizer of the coset H is exactly H, and $H \neq 1$.

It remains to prove that the stabilizers of two distinct points of Ω have trivial intersection. Let $xH, yH \in \Omega$ with $xH \neq yH$. We claim that

$$\mathrm{Stab}_G(xH) \cap \mathrm{Stab}_G(yH) = 1.$$

Indeed, for any $g \in G$, we have

$$g \in \mathrm{Stab}_G(xH) \iff g \cdot (xH) = xH \iff x^{-1}gx \in H \iff g \in H^{x^{-1}}.$$

Thus $\mathrm{Stab}_G(xH) = H^{x^{-1}}$, and similarly $\mathrm{Stab}_G(yH) = H^{y^{-1}}$. Therefore

$$\mathrm{Stab}_G(xH) \cap \mathrm{Stab}_G(yH) = H^{x^{-1}} \cap H^{y^{-1}} = \left(H \cap H^{y^{-1}x}\right)^{x^{-1}}.$$

Since $xH \neq yH$, we have $y^{-1}x \notin H$. Hence, by the Frobenius complement condition,

$$H \cap H^{y^{-1}x} = 1.$$

It follows that $\mathrm{Stab}_G(xH) \cap \mathrm{Stab}_G(yH) = 1$. Therefore the action is transitive but not regular, $|\Omega| > 1$, and the stabilizers of any two distinct points have trivial intersection. Hence the action of G on Ω is a Frobenius action. □

Next we illustrate Theorem 3.2.3 with the natural action of S_3 on the set $\{1, 2, 3\}$.

Example 3.2.4 (The natural action of S_3)**.** Let $G = S_3$ and consider the set $\Omega = \{1, 2, 3\}$. The group S_3 acts naturally on Ω by evaluation: for $\sigma \in S_3$ and $i \in \Omega$, we set

$$\sigma \cdot i = \sigma(i).$$

We show that this action is a Frobenius action. First, the action is transitive. Indeed, given $i, j \in \Omega$ with $i \neq j$, the transposition $(i\, j) \in S_3$ sends i to j.

Moreover, the action is not regular. Indeed, the stabilizer of the point 3 is

$$\mathrm{Stab}_G(3) = \langle (1\,2) \rangle \neq 1.$$

We now check the fixed point condition. The nonidentity elements of S_3 are the three transpositions

$$(1\,2), \qquad (1\,3), \qquad (2\,3),$$

and the two 3-cycles

$$(1\,2\,3), \qquad (1\,3\,2).$$

Each transposition fixes exactly one point:

$$(1\,2) \text{ fixes } 3, \qquad (1\,3) \text{ fixes } 2, \qquad (2\,3) \text{ fixes } 1.$$

On the other hand, the elements

$$(1\,2\,3), \qquad (1\,3\,2)$$

do not fix any element of Ω. Hence every nonidentity element of S_3 fixes at most one point. Therefore the natural action of S_3 on Ω is a Frobenius action.

By Theorem 3.2.3, the stabilizer of a point is a Frobenius complement. In particular, choosing the point 3, we obtain

$$H = \mathrm{Stab}_G(3) = \langle (1\,2) \rangle.$$

Moreover, the Frobenius kernel is the set of elements which fix no point, together with the identity. Hence

$$N = \{1, (1\,2\,3), (1\,3\,2)\} = \langle (1\,2\,3) \rangle.$$

Thus we recover the Frobenius group structure $S_3 = N \rtimes H$, where $N \cong C_3$ and $H \cong C_2$.

We conclude the section by presenting a more general construction of Frobenius

groups obtained from affine transformations over finite fields.

Example 3.2.5 (Affine group over a finite field)**.** Let $K = \mathbb{F}_q$ be the finite field with q elements, where $q = p^s$ for some prime number p and some integer $s \geq 1$, and assume that $q > 2$. Consider the set

$$G = \{f_{a,b} : K \to K \mid f_{a,b}(x) = ax + b,\ a \in K^\times,\ b \in K\},$$

where $K^\times = K \setminus \{0\}$ is the multiplicative group of the field. We endow G with the operation of composition of maps.

The identity element of G is $f_{1,0}$, that is, the map $x \mapsto x$. Moreover, if $f_{a,b} \in G$, then $f_{a,b}^{-1} = f_{a^{-1},-a^{-1}b}$, since $f_{a^{-1},-a^{-1}b}(x) = a^{-1}x - a^{-1}b$.

The group G acts naturally on K by evaluation: for $f_{a,b} \in G$ and $x \in K$, we set

$$f_{a,b} \cdot x = f_{a,b}(x) = ax + b.$$

We show that this action is a Frobenius action.

First, the action of G on K is transitive. Indeed, if $x, y \in K$, then the map

$$f_{1,y-x} : K \to K, \qquad f_{1,y-x}(z) = z + (y - x),$$

belongs to G and sends x to y.

The action of G on K is not regular. To see this, consider the stabilizer of 0. An element $f_{a,b} \in G$ fixes 0 if and only if $b = 0$. Therefore the stabilizer of 0 is

$$H = \{f_{a,0} \mid a \in K^\times\}.$$

Since $q > 2$, the group $K^\times$ has order $q-1 > 1$. Thus $H \neq 1$, and the action is not regular.

It remains to check that every nonidentity element of G fixes at most one point of K. Let $f_{a,b} \in G$. A fixed point of $f_{a,b}$ is an element $x \in K$ such that $f_{a,b}(x) = ax + b = x$. Equivalently, $(a - 1)x = -b$.
If $a \neq 1$, then $a - 1 \neq 0$, and this equation has exactly one solution:

$$x = \frac{-b}{a-1}.$$

If $a = 1$, then

$$f_{a,b}(x) = x + b.$$

When $b = 0$, $f_{a,b}$ is the identity map. When $b \neq 0$, the equation $x + b = x$ has no solution. Therefore every nonidentity element of G fixes either one point or no point.

Therefore G acts transitively and not regularly on K, and every nonidentity element of G fixes at most one point. Hence the action is a Frobenius action. By Theorem 3.2.3, G is a Frobenius group.

The Frobenius complement of G is the stabilizer of 0, namely

$$H = \{f_{a,0} \mid a \in K^\times\}.$$

The elements of G which fix no point of K are precisely the nonidentity translations, hence the Frobenius kernel is the set

$$N = \{f_{1,b} \mid b \in K\}.$$

Indeed, the identity map corresponds to the translation with $b = 0$, while the translations with $b \neq 0$ have no fixed points.

Thus $G = N \rtimes H$, where $N \cong (K, +)$ and $H \cong K^\times$. In particular,

$$|N| = q, \qquad |H| = q - 1,$$

and

$$|G| = q(q - 1).$$

Chapter 4

The Thompson Theorem

The purpose of this chapter is to investigate some of the main structural properties of Frobenius groups, with particular emphasis on the Frobenius kernel. More precisely, we will prove that the Frobenius kernel is always a nilpotent group. This is one of the deepest structural properties of Frobenius groups and was first proved by J. Thompson in his doctoral thesis [Tho59].

The proof will proceed in several steps. We first establish the result under the additional assumption that the kernel is solvable. Then we remove this hypothesis and pass to the general case by using a deep theorem of Thompson. Finally, we briefly discuss a stronger version of Thompson's result and some of its consequences.

The exposition in this chapter follows mainly Isaacs' treatment of Frobenius actions and the Thompson subgroup in Chapters 6 and 7 of [Isa08].

4.1 Solvable Frobenius kernels

We begin with the case in which the Frobenius kernel is assumed to be solvable. We first present a group-theoretic result that will play a crucial role in the proof of the nilpotency of solvable Frobenius kernels.

Lemma 4.1.1. *Let Π be a partition of a finite group G, and suppose that G acts by automorphisms on a finite abelian group V. If V has an element whose order does not divide $|\Pi| - 1$, then there exists a member $X \in \Pi$ such that*

$$C_V(X) > 1.$$

Proof. Assume that $C_V(X) = 1$ for every $X \in \Pi$.

For every subgroup $H \leq G$ and every element $v \in V$, define

$$v_H = \prod_{h \in H} v^h.$$

This element v_H of V is well-defined because V is abelian, and therefore the order of the factors does not matter.

We first observe that, for every $H \leq G$, the element v_H is fixed by H. Indeed, if $k \in H$, then

$$(v_H)^k = \prod_{h \in H} (v^h)^k = \prod_{h \in H} v^{hk}.$$

As h runs through H, also hk runs through H. Hence the last product has the same factors as v_H, only possibly in a different order. Since V is abelian, we conclude that $(v_H)^k = v_H$, as claimed. Thus $v_H \in C_V(H)$. In particular, if $X \in \Pi$, then $v_X \in C_V(X)$. By our assumption $C_V(X) = 1$, we obtain $v_X = 1$ for every $X \in \Pi$ and every $v \in V$.

Now fix $v \in V$ and multiply these equalities over all members of the partition Π. Since each v_X is equal to 1, we have

$$1 = \prod_{X \in \Pi} v_X = \prod_{X \in \Pi} \prod_{x \in X} v^x. \tag{4.1.1}$$

We now rewrite the right-hand side of (4.1.1). Since Π is a partition of G, every nonidentity element of G belongs to exactly one member of Π. Therefore, in the double product each factor v^g, with $g \in G \setminus \{1\}$, occurs exactly once. On the other hand, the identity element $1 \in G$ belongs to every subgroup $X \in \Pi$. Hence the factor $v^1 = v$ occurs once for each member of Π, that is, $|\Pi|$ times.

Thus the double product contains all the factors occurring in

$$v_G = \prod_{g \in G} v^g$$

and, in addition, it contains $|\Pi| - 1$ extra copies of the factor $v = v^1$. Since V is abelian, we may rearrange the factors and obtain

$$\prod_{X \in \Pi} \prod_{x \in X} v^x = v_G v^{|\Pi|-1}. \tag{4.1.2}$$

Combining (4.1.2) with (4.1.1), we get

$$1 = v_G v^{|\Pi|-1} \tag{4.1.3}$$

Choose $v \in V$ such that $v^{|\Pi|-1} \neq 1$, which is possible by hypothesis. Then the equality above gives $v_G \neq 1$. Since every member $X \in \Pi$ is a subgroup of G, we have

$$v_G \in C_V(G) \leq C_V(X).$$

Thus $v_G \in C_V(X)$ for every $X \in \Pi$. Since $v_G \neq 1$, this shows that there exists some $X \in \Pi$ such that

$$C_V(X) > 1.$$

□

As observed in Definition 2.3.3, Frobenius groups provide a natural example of partition. Indeed, if G is a Frobenius group with Frobenius kernel N and complement H, then the collection consisting of N and of all distinct conjugates of H is the Frobenius partition of G. We are now ready to prove the main result of this section. It shows that the nilpotency of the Frobenius kernel can be obtained under the additional assumption that the kernel is solvable. The solvable case is already contained in Higman's work on groups admitting fixed-point-free automorphisms of prime order [Hig57]. In the following proof, we follow the presentation given in [Isa08, Chapter 6].

Theorem 4.1.2. *Let N be a solvable Frobenius kernel. Then N is nilpotent.*

Proof. Since N is a Frobenius kernel, there exists a nontrivial group H acting on N by automorphisms in such a way that the action is Frobenius. In other words,

$$C_N(h) = 1 \quad \text{for every } h \in H \setminus \{1\}.$$

By Cauchy's Theorem 1.2.14, H contains a subgroup of prime order p, say $H_0 \leq H$ with $|H_0| = p$ for some prime p. By Corollary 3.1.7, the restriction of a Frobenius action to a subgroup of H is still Frobenius. Thus H_0 also acts on N as a Frobenius group of automorphisms.

Replacing H by H_0, we may therefore assume, without loss of generality, that

$$|H| = p$$

is prime. We now form the semidirect product $G = N \rtimes H$. Thus $N \lhd G$, $H \leq G$, and the action of H on N is the given Frobenius action.

If $N = 1$, then there is nothing to prove. Suppose, by contradiction, that $1 \neq N$ is not nilpotent. We argue by induction on $|N|$ and assume that N is a counterexample of minimal possible order. This assumption will allow us to apply induction whenever we obtain a solvable Frobenius kernel of order strictly smaller than $|N|$.

Since $N \neq 1$, we can choose a nontrivial subgroup $U \leq N$ which is minimal normal in G. Such a subgroup exists because $N \triangleleft G$, so the set of nontrivial normal subgroups of G contained in N is nonempty.

We now consider the quotient N/U. Since $U \triangleleft G$, in particular $U \triangleleft N$. Moreover, U is H-invariant: indeed, $H \leq G$, and normality of U in G implies that $U^h = U$ for every $h \in H$.

Therefore the action of H on N induces an action of H on N/U. By Corollary 3.1.8, this induced action is again Frobenius. Hence N/U is again a Frobenius kernel.

Furthermore, N/U is solvable because it is a quotient of the solvable group N. Since $|N/U| < |N|$, the inductive hypothesis of N implies that N/U is nilpotent.

By the Definition 1.2.12 of the nilpotent residual, and since N is not nilpotent by assumption, we have

$$\gamma_\infty(N) \neq 1.$$

Since N/U is nilpotent and $U \triangleleft N$, the definition of $\gamma_\infty(N)$ gives $\gamma_\infty(N) \leq U$.

Moreover, by Lemma 1.2.13, $\gamma_\infty(N)$ is characteristic in N. Since $N \triangleleft G$, it follows that $\gamma_\infty(N) \triangleleft G$.

Since U was chosen minimal normal in G, and $1 \neq \gamma_\infty(N) \leq U$, we necessarily obtain

$$\gamma_\infty(N) = U.$$

Therefore U is the unique minimal normal subgroup of G contained in N. Indeed, let $V \leq N$ be another minimal normal subgroup of G. Repeating the same argument with V in place of U, we obtain that N/V is nilpotent. Hence, by the definition of $\gamma_\infty(N)$, $\gamma_\infty(N) \leq V$. Since V is minimal normal in G and $\gamma_\infty(N) \neq 1$, it follows that $V = \gamma_\infty(N)$. Hence $V = U$.

Since U is a solvable minimal normal subgroup of G, Proposition 1.2.17 implies that U is elementary abelian. Thus there exists a prime q and an integer $m \geq 1$ such that $U \cong C_q^m$. In particular, U is an abelian q-group.

Since N is not nilpotent, it cannot be a q-group. Hence there exists a prime $r \neq q$ dividing $|N|$, and therefore N contains a nontrivial Sylow r-subgroup R. We may choose R to be H-invariant. Indeed, since H acts on N by automorphisms, it permutes the Sylow r-subgroups of N. As $|H| = p$ is prime, every orbit of this action has length either 1 or p. Since the action of H on N is Frobenius, the orders of H and N are coprime. As $|H| = p$, we have $p \nmid |N|$. It follows that p cannot divide any divisor of $|N|$, and therefore

$$p \nmid |\operatorname{Syl}_r(N)|.$$

Consequently, not all orbits can have length p, and therefore there exists an orbit of length

1. This means that some Sylow r-subgroup R satisfies

$$R^h = R$$

for every $h \in H$. Hence R is H-invariant, as claimed above.

We now show that R is not normal in N. Suppose, by contradiction, that $R \lhd N$. Since all Sylow r-subgroups of N are conjugate, a normal Sylow r-subgroup must be unique. Hence R is the unique Sylow r-subgroup of N, and therefore R is characteristic in N. Since $N \lhd G$, every characteristic subgroup of N is normal in G. Thus $R \lhd G$. Now $R \leq N$ and $R \neq 1$, so R contains a minimal normal subgroup of G, say V. Since $V \leq R \leq N$, the uniqueness of U implies that $V = U$. Hence $U \leq R$. But U is a q-group, while R is an r-group with $r \neq q$. This is impossible. Therefore

$$R \text{ is not normal in } N, \tag{4.1.4}$$

as claimed.

We now consider the subgroup of N generated by R and U, $RU = \langle R, U \rangle$. Both R and U are H-invariant, and therefore RU is H-invariant as well. Hence H acts on RU. This action is Frobenius, because for every $h \in H \setminus \{1\}$ we have $C_{RU}(h) \leq C_N(h) = 1$. In particular, RU is again a Frobenius kernel.

Furthermore, RU is normal in N. To see this, consider the quotient RU/U. Since $U \lhd N$, RU/U is a Sylow r-subgroup of N/U. But N/U is nilpotent, and therefore all its Sylow subgroups are normal. Hence $RU/U \lhd N/U$.
It follows that

$$RU \text{ is normal in } N. \tag{4.1.5}$$

We now prove that $RU = N$. Suppose, by contradiction, that $RU < N$. Since RU is H-invariant and the action of H on RU is Frobenius, the group RU is a solvable Frobenius kernel. Moreover, $|RU| < |N|$. Therefore, by the inductive hypothesis, the group RU is nilpotent. In particular, every Sylow subgroup of RU is normal in RU. Since R is the unique Sylow r-subgroup of RU, we obtain that R is characteristic in RU and so by (4.1.5) $R \lhd N$, that contradicts (4.1.4).

We now consider the product RH. Since R is H-invariant, RH is a subgroup of G. Moreover, the action of H on R is Frobenius, because $R \leq N$ and the action of H on N is Frobenius. Therefore RH is a Frobenius group with Frobenius kernel R and Frobenius complement H.

The Frobenius partition of RH is given by

$$\Pi = \{R\} \cup \{H^x \mid x \in R\}.$$

Since the number of conjugates of H in RH is $|R|$, we have $|\Pi| = 1+|R|$. Thus $|\Pi|-1 = |R|$, which is a power of r.

The order of every nonidentity element of U is a power of q, and therefore it does not divide $|R| = |\Pi| - 1$. Thus by Lemma 4.1.1, there exists a member $X \in \Pi$ such that

$$C_U(X) > 1.$$

Now every member of Π is either R or a conjugate of H in RH. We claim that the subgroup X given by Lemma 4.1.1 must be R. Indeed, for every $x \in R$, the subgroup H^x is a conjugate of H. Since the action of H on U is Frobenius, also the action of each conjugate H^x on U is Frobenius. Hence $C_U(H^x) = 1$ for every $x \in R$. Therefore X cannot be one of the conjugates H^x, because we know that $C_U(X) > 1$. Consequently,

$$X = R.$$

Thus

$$C_U(R) > 1$$

and there exists a nontrivial element $u \in U$ which commutes with every element of R.

Moreover,

$$1 < C_U(R) \leq Z(RU).$$

Indeed, if $u \in C_U(R)$, then u commutes with every element of R. Since U is abelian, u also commutes with every element of U. Therefore u commutes with every element of $N = RU$, and hence $u \in Z(RU) = Z(N)$. Thus $Z(N)$ is nontrivial. Moreover, $Z(N)$ is normal in G, because $N \triangleleft G$ and $Z(N)$ is characteristic in N. By the uniqueness of U as the minimal normal subgroup of G contained in N, we must have

$$U \leq Z(N).$$

Therefore U is central in $N = RU$, and in particular U centralizes R.

Now $U \triangleleft N$, and R is a Sylow r-subgroup of $N = RU$. Since U centralizes R, conjugation by elements of U fixes R, while conjugation by elements of R clearly preserves R. Hence $R \triangleleft RU = N$. This contradicts (4.1.4) that is R is not normal in N. The contradiction proves that our original assumption was false. Therefore N must be nilpotent, as desired. $\square$

4.2 Thompson's Theorem: the general case

We now remove the solvability assumption on the Frobenius kernel and prove the result in full generality. The key ingredient will be the result of Thompson's doctoral thesis concerning a criterion for the existence of a normal p-complement, in a finite group, where p is an odd prime.

Recall that a subgroup $K \triangleleft G$ is called a normal p-complement of G if $p \nmid |K|$ and $|G : K|$ is a power of p.

For example, if G is nilpotent, then G has a normal p-complement for every prime p dividing $|G|$. Indeed, a finite nilpotent group is the direct product of its Sylow subgroups. If $P \in \mathrm{Syl}_p(G)$, then

$$G = P \times \prod_{q \neq p} P_q,$$

where $P_q \in \mathrm{Syl}_q(G)$. The subgroup

$$K = \prod_{q \neq p} P_q$$

is normal in G, its order is not divisible by p, and the index $|G : K|$ is a power of p. Hence K is a normal p-complement of G.

A fundamental tool for studying normal p-complements is Frobenius' normal p-complement theorem. This result gives a criterion for the existence of a normal p-complement in terms of the normalizers of the nonidentity p-subgroups, the so called p-local subgroups of a group G. We state it in the form given in [Isa08, Theorem 5.26].

Theorem 4.2.1 (Frobenius' normal p-complement theorem)**.** *Let G be a finite group and let p be a prime. Then the following conditions are equivalent:*

1. *G has a normal p-complement.*
2. *$N_G(X)$ has a normal p-complement for every nonidentity p-subgroup $X \leq G$.*
3. *$N_G(X)/C_G(X)$ is a p-group for every p-subgroup $X \leq G$.*

This theorem shows that the existence of a normal p-complement in G is determined by the local structure of its p-subgroups. However, verifying the condition for every p-subgroup can be difficult, since a finite group may contain many such subgroups.

Thompson considerably strengthened this criterion by proving that, for $p \neq 2$, it is sufficient to restrict the attention to certain characteristic subgroups of a fixed Sylow p-subgroup of G. This refinement will be the key ingredient in the proof of the nilpotency of Frobenius kernels in the general case.

The following result is stated as [Isa08, Theorem 6.23].

Theorem 4.2.2 (Thompson)**.** *Let $P \in \mathrm{Syl}_p(G)$, where G is a finite group and $p \neq 2$. Assume that $N_G(X)$ has a normal p-complement for every nonidentity characteristic subgroup X of P. Then G has a normal p-complement.*

The condition $p \neq 2$ is essential. Indeed, the statement is false in general for $p = 2$. A standard example is given by $G = S_4$. Let $P \in \mathrm{Syl}_2(S_4)$. Then $P \cong D_8$. Let X be a nontrivial characteristic subgroup of P. Since X is characteristic in P, it is in particular normal in P. Hence, as P is a 2-group, we have $X \cap Z(P) \neq 1$. Moreover, $Z(P)$ has order 2, and therefore $Z(P) \leq X$. Hence

$$N_{S_4}(X) \leq N_{S_4}(Z(P)) = P.$$

Thus $N_{S_4}(X)$ is a 2-group and therefore it has a normal 2-complement, namely the trivial subgroup. However, S_4 has no normal 2-complement. Indeed, such a subgroup would have order 3. A normal subgroup of order 3 would contain all the conjugates of a 3-cycle, which is impossible since S_4 contains eight 3-cycles. Therefore the conclusion of Thompson's theorem fails for $p = 2$.

Theorem 4.2.2 is already a strong refinement of Frobenius' normal p-complement Theorem 4.2.1, since it restricts the verification to the nontrivial characteristic subgroups of a fixed Sylow p-subgroup P. Thompson later obtained an even sharper criterion. As we will see later, this result shows that it is enough to consider only two distinguished characteristic subgroups of P: its center $Z(P)$ and the so-called *Thompson subgroup* $J(P)$.

Before introducing the Thompson subgroup, we define a set of elementary abelian subgroups that will be used in its construction.

Let P be a finite p-group. Denote by $m(P)$ the largest integer m such that P contains an elementary abelian subgroup of order p^m. We define

$$E(P) = \{\, A \leq P \mid A \text{ is elementary abelian and } |A| = p^{m(P)} \,\}.$$

Thus $E(P)$ is the set of all elementary abelian subgroups of P having maximal possible order.

Definition 4.2.3. Let P be a finite p-group. The *Thompson subgroup* of P is the subgroup generated by all elementary abelian subgroups of maximal order contained in P. More precisely,

$$J(P) = \langle A \mid A \in E(P) \rangle.$$

We first observe two simple but important properties of $J(P)$.

First, if $P \neq 1$, then

$$J(P) \neq 1.$$

Indeed, every nontrivial p-group contains an element of order p. Therefore P contains at least one nontrivial elementary abelian subgroup, and so $E(P)$ is nonempty. It follows that the subgroup generated by the members of $E(P)$ is nontrivial.

Moreover, $J(P)$ is characteristic in P. To see this, let $\varphi \in \mathrm{Aut}(P)$. Since automorphisms preserve orders and the property of being elementary abelian, φ sends elementary abelian subgroups of maximal order to elementary abelian subgroups of maximal order. Hence φ permutes the members of $E(P)$. Consequently, it preserves the subgroup generated by all of them, that is,

$$\varphi(J(P)) = J(P).$$

Thus

$$J(P) \,\mathrm{char}\, P.$$

We now illustrate the construction of $J(P)$ with a simple example.

Example 4.2.4 (The Thompson subgroup of D_8)**.** Consider

$$P = D_8 = \langle r, s \mid r^4 = s^2 = 1,\ srs = r^{-1} \rangle,$$

the dihedral group of order 8.

Since $|D_8| = 8 = 2^3$, the only elementary abelian subgroups that can occur in D_8 are elementary abelian 2-groups.

The elementary abelian subgroups of maximal order in D_8 are obtained from the central involution r^2 together with a reflection. More precisely, since r^2 is central and every reflection has order 2, the subgroups

$$A_1 = \langle r^2, s \rangle = \{1, r^2, s, r^2 s\}$$

and

$$A_2 = \langle r^2, rs \rangle = \{1, r^2, rs, r^3 s\}$$

are elementary abelian of order 4.

There is no elementary abelian subgroup of order 8. Indeed, such a subgroup would have to be the whole group D_8, but D_8 is not abelian. Hence the maximal possible order of an elementary abelian subgroup of D_8 is 4.

Therefore

$$E(P) = \{A_1, A_2\}.$$

By definition, the Thompson subgroup is generated by the members of $E(P)$. Hence

$$J(P) = \langle A_1, A_2 \rangle.$$

Since A_1 contains s and A_2 contains rs, the subgroup generated by A_1 and A_2 contains both s and rs. It follows that it also contains $(rs)s = r$. Thus $J(P)$ contains both r and s, and therefore $J(P) = D_8$.

We are now ready to prove the main result of this chapter: every Frobenius kernel is nilpotent. This classical theorem is due to Thompson [Tho59]; for the presentation followed here, see also [Isa08, Theorem 6.24].

Theorem 4.2.5. *Let N be a Frobenius kernel. Then N is nilpotent.*

Proof. We proceed by induction on $|N|$. If $|N| = 1$, then there is nothing to prove. Thus we may assume that $|N| > 1$, and that the result holds for all Frobenius kernels of order strictly smaller than $|N|$.

Since N is a Frobenius kernel, there exists a nontrivial group H acting on N by automorphisms in such a way that the action is Frobenius. As in the proof of Theorem 4.1.2, by replacing H with a subgroup of prime order, we may assume that $|H| = p$ for some prime p. We first consider the case where there exists a subgroup $M \leq N$ such that $1 < M < N$ with $M \lhd N$, and such that M is H-invariant. Then H acts on M, and this action is still Frobenius. Indeed, if $h \in H \setminus \{1\}$, then

$$C_M(h) \leq C_N(h) = 1.$$

Moreover, since $M \lhd N$ and M is H-invariant, the action of H also induces a Frobenius action on the quotient N/M, by Corollary 3.1.8.

Thus both M and N/M are Frobenius kernels. Since

$$|M| < |N| \quad \text{and} \quad |N/M| < |N|,$$

inductive hypothesis implies that both M and N/M are nilpotent. In particular, both M and N/M are solvable. Therefore N is solvable. By Theorem 4.1.2, N is nilpotent and the result holds.

We can therefore assume that no such subgroup M exists in N. In other words, we assume that N contains no nontrivial proper H-invariant normal subgroup.

We claim that, under this assumption, N must be an r-group for some prime r. Once this is proved, the conclusion follows immediately, since our r-group is obviously nilpotent.

Suppose, by contradiction, that N is not an r-group for any prime r. Then $|N|$ is divisible by at least two distinct primes. Hence at least one odd prime divides $|N|$; choose such a prime and denote it by r. Let

$$R \in \mathrm{Syl}_r(N).$$

As in the proof of Theorem 4.1.2, we may choose R to be H-invariant, because H permutes the Sylow r-subgroups of N and $(|H|, |N|) = 1$.

We now want to apply Thompson's Theorem 4.2.2 to the group N and to its Sylow r-subgroup R. Let

$$1 \neq X \operatorname{char} R.$$

Since R is H-invariant, the subgroup X is H-invariant as well. Indeed, the action of each element of H restricts to an automorphism of R, and X is fixed by every automorphism of R. We first observe that X is a proper subgroup of N. In fact, $X \leq R$, and R is an r-group. If $X = N$, then N would be an r-group, contrary to our assumption. Hence

$$X < N.$$

Moreover, X is not normal in N. Otherwise, X would be a nontrivial proper normal subgroup of N which is also H-invariant. This is impossible in the case we are considering. Therefore X is not normal in N. Equivalently,

$$N_N(X) < N.$$

Since X is H-invariant, its normalizer $N_N(X)$ is H-invariant as well. Indeed, every element of H induces an automorphism of N preserving X, and therefore also preserves the set of elements normalizing X. The action of H on $N_N(X)$ is Frobenius. Indeed, $N_N(X)$ is H-invariant and

$$C_{N_N(X)}(h) \leq C_N(h) = 1$$

for every $h \in H \setminus \{1\}$. Moreover, $N_N(X) < N$. Thus, by the inductive hypothesis, the group $N_N(X)$ is nilpotent.

As observed above, every finite nilpotent group has a normal r-complement. Hence $N_N(X)$ has a normal r-complement for every nontrivial characteristic subgroup X of R.

By applying Theorem 4.2.2 to the group N, with $R \in \mathrm{Syl}_r(N)$, it follows that N has a normal r-complement, say K.

The subgroup K is nontrivial and proper in N. Indeed, r divides $|N|$, while r does

not divide $|K|$. Moreover, since N is not an r-group, the subgroup K is nontrivial.

We now observe that K is characteristic in N. Indeed, K is the unique normal r-complement of N, and every automorphism of N preserves the properties of being normal, having order not divisible by r, and having index a power of r. Therefore every automorphism of N must fix K, and hence $K \operatorname{char} N$.

Since H acts on N by automorphisms, it follows in particular that K is H-invariant. Thus K is a nontrivial proper H-invariant normal subgroup of N, contradicting the assumption of the case we are considering.

This contradiction shows that N must be an r-group for some prime r. Hence N is nilpotent, and the theorem is proved. □

After his work on Frobenius kernels, Thompson obtained a stronger normal p-complement criterion. In 1964, he proved a sharper version of the Theorem 4.2.2, with a shorter proof [Tho64].

Theorem 4.2.6 (Thompson)**.** *Let $P \in \mathrm{Syl}_p(G)$, where G is a finite group and $p \neq 2$. Assume that $C_G(Z(P))$ has a normal p-complement and that $N_G(J(P))$ has a normal p-complement. Then G has a normal p-complement.*

This theorem implies Theorem 4.2.2. Indeed, assume that $N_G(X)$ has a normal p-complement for every nonidentity characteristic subgroup X of P. Since $P \neq 1$, both $Z(P)$ and $J(P)$ are nontrivial characteristic subgroups of P. Hence $N_G(Z(P))$ and $N_G(J(P))$ have normal p-complements. Moreover, $C_G(Z(P))$ is a normal subgroup of $N_G(Z(P))$, and therefore it also has a normal p-complement. Thus the hypotheses of Theorem 4.2.6 are satisfied, and so G has a normal p-complement.

Although we do not present the complete proof of Theorem 4.2.6, we briefly describe its main idea. A complete proof can be found in [Isa08, Chapter 7, Theorem 7.1].

Before sketching the proof, we recall the notion of p-solvability. A finite group G is said to be *p-solvable* if it has a normal series $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_m = G$ such that each factor G_i/G_{i-1} is either a p-group or a p'-group, that is, a group whose order is not divisible by p.

Sketch of the proof of Theorem 4.2.6. The argument proceeds by contradiction, assuming that G is a counterexample of minimal order. The first part of the proof is devoted to showing that such a counterexample must satisfy very restrictive structural properties. In particular, one proves that G is p-solvable and that it has no nontrivial normal subgroup of order prime to p, that is $O_{p'}(G) = 1$. Moreover, one obtains $C_G(Z(P)) = P$. These reductions allow one to show that $J(P) \lhd G$. Consequently, $G = N_G(J(P))$. By hypothesis $N_G(J(P))$ has a normal p-complement, and therefore G itself has a normal

p-complement. This contradicts the choice of G as a counterexample. Hence no such counterexample exists, and the theorem follows. □

Theorem 4.2.6 is sharper than Theorem 4.2.2 in two respects. First, it reduces the verification to the two subgroups $Z(P)$ and $J(P)$. Second, in the case of the center, it is enough to consider the centralizer $C_G(Z(P))$, rather than the whole normalizer $N_G(Z(P))$.

Beyond its role in the study of Frobenius kernels, the Thompson subgroup and the normal p-complement criterion have further important applications in finite group theory. In particular, they provide a key ingredient in the group-theoretic proof of Burnside's p^aq^b-theorem, avoiding the use of character theory.

We conclude this section with an example of a Frobenius group whose kernel is nilpotent but not abelian. The same construction also shows that Frobenius kernels can have arbitrarily large nilpotency class.

Example 4.2.7. Let $\mathbb{F}_{11}$ be the field with 11 elements and consider the group $K = IT_3(11)$, consisting of all upper unitriangular 3×3 matrices over $\mathbb{F}_{11}$, that is,

$$K = \left\{ X(t,u,v) = \begin{pmatrix} 1 & t & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \middle| \, t,u,v \in \mathbb{F}_{11} \right\}.$$

The order of K is $|K| = 11^3$, since each element of K is uniquely determined by the three entries $t, u, v \in \mathbb{F}_{11}$. The group K is not abelian.
Moreover, K is nilpotent of class 2. Indeed, a direct computation of commutators shows that every commutator lies in the subgroup $Z = \{X(0,u,0) \mid u \in \mathbb{F}_{11}\}$. This subgroup is contained in the center of K, because its elements commute with all matrices of the form $X(t,u,v)$. Hence

$$[K,K] \leq Z(K).$$

Therefore

$$\gamma_3(K) = [[K,K],K] = 1.$$

Since K is not abelian, we have $[K,K] \neq 1$. Thus the nilpotency class of K is exactly 2.

We now construct a group of automorphisms acting fixed-point-freely on K. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in GL_3(11),$$

and set

$$H = \langle A \rangle.$$

In the field $\mathbb{F}_{11}$ we have $3^5 = 1$ and $4^5 = 1$, while $3^m \neq 1$ and $4^m \neq 1$ for every $0 < m < 5$. Hence $A^5 = I_3$, and no smaller positive power of A is the identity. Therefore $H = \langle A \rangle$ is a cyclic group of order 5.

The group H acts on K by conjugation. We claim that this action is fixed-point-free. Let $0 < m < 5$. A direct computation gives

$$A^m X(t,u,v) A^{-m} = X(3^{-m}t, 4^{-m}u, 5^{-m}v).$$

Suppose now that $X(t,u,v)$ is fixed by conjugation by A^m. Then $A^m X(t,u,v)A^{-m} = X(t,u,v)$, and therefore $3^{-m}t = t$, $4^{-m}u = u$, and $5^{-m}v = v$. Since 3^{-m}, 4^{-m}, and 5^{-m} are all different from 1 for $0 < m < 5$, it follows that $t = u = v = 0$. Thus the only element of K fixed by A^m is the identity matrix. Therefore every nonidentity element of H acts fixed-point-freely on K. By Theorem 3.1.1, the semidirect product

$$G = K \rtimes H$$

is a Frobenius group, with Frobenius kernel K and Frobenius complement H.

In particular, this example gives a Frobenius group of order

$$|G| = |K||H| = 11^3 \cdot 5 = 6655.$$

Its Frobenius kernel is not abelian, but it is nilpotent of class 2.

Remark 4.2.8. The previous example is the first case of a more general construction. Let $n \geq 3$ and let q be a prime such that $q > n$. Choose a prime p such that $q \mid p - 1$, and work over the field $\mathbb{F}_p$. Let

$$K = IT_n(p),$$

the group of all upper unitriangular $n \times n$ matrices over $\mathbb{F}_p$. The same construction as above, using a suitable cyclic group of diagonal matrices of order q, shows that K occurs as the Frobenius kernel of a Frobenius group. $IT_n(p)$ is nilpotent of class $n-1$. Thus, as n increases, the nilpotency class of the Frobenius kernel K also increases. In particular, Frobenius kernels can have arbitrarily large nilpotency class.

4.3 Uniqueness of the Frobenius partition

After proving the nilpotency of Frobenius kernels, we can derive some further structural properties of Frobenius groups. In particular, this result allows us to complete the discussion on Frobenius partitions started in Chapter 2.

We first prove a useful property concerning the normal subgroups of a Frobenius group, which will be the key ingredient in establishing the uniqueness of the Frobenius partition. We follow the presentation of [Hup25, Theorems 8.16 and 8.17].

Lemma 4.3.1. *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. If $M \lhd G$, then either $M \leq N$ or $N < M$.*

Proof. Suppose that M is not contained in N. Then there exists an element $m \in M$ such that $m \notin N$. By the Frobenius partition 2.3.3,

$$G = N \cup \bigcup_{g \in G} \big(H^g \setminus \{1\}\big),$$

and so the element m lies in a conjugate of H. Hence there exist $g \in G$ and $h \in H \setminus \{1\}$ such that $m = g^{-1}hg$. Since M is normal in G and $m \in M$, it follows that

$$h = gmg^{-1} \in M.$$

Thus $M \cap H$ contains a nonidentity element h.

Since G is a Frobenius group with Frobenius kernel N and Frobenius complement H, the action of H on N by conjugation is fixed-point-free. In particular, the automorphism of N induced by h^{-1} , namely

$$\alpha_{h^{-1}} : N \longrightarrow N, \qquad \alpha_{h^{-1}}(n) = h^{-1}nh,$$

is fixed-point-free.

By Proposition 1.2.5 the map

$$\mu : N \longrightarrow N, \qquad \mu(n) = n^{-1}\alpha_{h^{-1}}(n) = n^{-1}h^{-1}nh,$$

is bijective. Therefore, for every $n \in N$, we have $\mu(n) = n^{-1}h^{-1}nh$. Since $h \in M$, also $h^{-1} \in M$. Moreover, M is normal in G, so $n^{-1}h^{-1}n \in M$. Therefore

$$\mu(n) = n^{-1}h^{-1}nh \in M.$$

Thus $\mu(N) \subseteq M$. But μ is bijective from N onto N, so $\mu(N) = N$. Hence $N \leq M$.

Moreover, $h \in M \cap H$ and $h \neq 1$, while $N \cap H = 1$. Hence $h \notin N$, and therefore $M \neq N$. We conclude that

$$N < M.$$

□

We can now apply this result to prove that the Frobenius partition associated with a Frobenius group is uniquely determined.

Theorem 4.3.2. *Every Frobenius group has exactly one Frobenius partition.*

Proof. Let G be a Frobenius group. Suppose that G has two Frobenius complements H_1 and H_2, with corresponding Frobenius kernels N_1 and N_2, respectively. Thus, by Theorem 2.3.1

$$G = N_1H_1 = N_2H_2,$$

with $N_1 \cap H_1 = 1$ and $N_2 \cap H_2 = 1$. By Lemma 4.3.1, applied to the Frobenius group G with kernel N_2 and to the normal subgroup $N_1 \lhd G$, we have either $N_1 \leq N_2$ or $N_2 < N_1$. Assume that $N_1 \leq N_2$ and for contradiction that $N_1 < N_2$. We are going to show that we must have $N_1 = N_2$. Indeed, since $G = N_1H_1$, every element of N_2 can be written as a product of an element of N_1 and an element of H_1. Therefore

$$N_2 = N_1(N_2 \cap H_1). \tag{4.3.1}$$

Moreover, $N_2 \cap H_1$ must be nontrivial. Indeed, if $N_2 \cap H_1 = 1$, then the equality (4.3.1) would give $N_2 = N_1$, contrary to the assumption $N_1 < N_2$.

By Proposition 3.1.6, the orders of N_1 and H_1 are coprime. Hence the orders of N_1 and $N_2 \cap H_1$ are also coprime. On the other hand, by Theorem 4.2.5, the Frobenius kernel N_2 is nilpotent. Therefore N_2 is the direct product of its Sylow subgroups. As $|N_1|$ and $|N_2 \cap H_1|$ are coprime, the prime divisors of $|N_1|$ and of $|N_2 \cap H_1|$ are disjoint. Hence the elements of N_1 and those of $N_2 \cap H_1$ belong to different Sylow components of N_2, and therefore commute. Thus $[N_1, N_2 \cap H_1] = 1$ and so every element of $N_2 \cap H_1$ centralizes N_1. This is impossible because $N_2 \cap H_1$ is a nontrivial subgroup of H_1 that acts on the Frobenius kernel N_1 by fixed-point-free automorphisms. Thus no nonidentity element of H_1 can centralize a nonidentity element of N_1. Thus we must have $N_1 = N_2$.

Next we prove that the two complements H_1 and H_2 are conjugate in G. Since $N_1 = N_2$, both H_1 and H_2 are complements of the same normal subgroup N_1 in G. Moreover, by Proposition 3.1.6, we have

$$\big(|N_1|, |G/N_1|\big) = 1.$$

Finally, N_1 is nilpotent by Theorem 4.2.5, and hence it is solvable. Therefore the hypotheses of Schur–Zassenhaus' Theorem 1.2.18 are satisfied and H_1 and H_2 are conjugate in G.

Consequently, the set of conjugates of H_1 is equal to the set of conjugates of H_2. Since the Frobenius kernels are also equal, the following two Frobenius partitions coincide:

$$G = N_1 \cup \bigcup_{g \in G} \left(H_1^g \setminus \{1\} \right) = N_2 \cup \bigcup_{g \in G} \left(H_2^g \setminus \{1\} \right)$$

showing the uniqueness of the Frobenius partition. □

Chapter 5

Further properties of Frobenius groups

Although a Frobenius group has the form of a semidirect product, the fixed-point-free action of the complement on the kernel imposes strong restrictions on both factors of the decomposition. For the kernel, the main structural result is Thompson's theorem. It is natural to ask whether analogous restrictions also hold for the complement H.

The examples 3.2.5 and 2.3.4 suggest that the complements occurring in Frobenius groups have a special form. In the affine group over a finite field $K = \mathbb{F}_q$, the Frobenius complement is isomorphic to $K^\times$. Since the multiplicative group of a finite field is cyclic, all Sylow subgroups of the complement are cyclic. In the group $S_3 \simeq C_3 \rtimes C_2$, the Frobenius complement itself is cyclic, being isomorphic to C_2.

This phenomenon is not accidental. Although a Frobenius complement need not itself be cyclic, Burnside's Theorem shows that the structure of its Sylow subgroups is highly restricted.

Theorem 5.0.1 (Burnside)**.** *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. If $P \in \mathrm{Syl}_p(H)$, then P is cyclic for $p > 2$, while for $p = 2$ it is either cyclic or generalized quaternion.*

We will not include the proof of this theorem; the interested reader can find it in [Hup25, Theorem 8.7(b)]. Nevertheless, its meaning is important for the structure theory of Frobenius groups. A general finite p-group can be structurally quite complicated. Burnside's theorem shows that this freedom disappears for the Sylow subgroups of a Frobenius complement: for odd primes they must be cyclic, and for $p = 2$ the only further possibility is the generalized quaternion case.

It is important, however, not to confuse this local restriction with a global description of the complement. Theorem 5.0.1 does not imply that the complement H itself is nilpotent. In fact, Burnside conjectured that Frobenius complements should be nilpotent, but this is false: a Frobenius complement need not even be solvable, as the next example shows [Hup25, Remark 8.8(b)].

Example 5.0.2. Let p be a prime such that $p^2 \equiv 1 \pmod 5$. Let $V = \mathbb{F}_p^2$ be the elementary abelian group, written additively. We also consider the special linear group

$$\mathrm{SL}_2(p) = \{A \in \mathrm{GL}_2(p) \mid \det(A) = 1\}.$$

This group acts naturally on V by matrix multiplication: if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(p) \qquad \text{and} \qquad v = \begin{pmatrix} x \\ y \end{pmatrix} \in V,$$

then

$$A \cdot v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Since every element of $\mathrm{SL}_2(p)$ is invertible, this gives an action by automorphisms of the additive group V. For odd p, the center of $\mathrm{SL}_2(p)$ is $Z(\mathrm{SL}_2(p)) = \{I, -I\}$. The projective special linear group $\mathrm{PSL}_2(p)$ is defined as the quotient

$$\mathrm{PSL}_2(p) = \mathrm{SL}_2(p)/Z(\mathrm{SL}_2(p)).$$

Let

$$\pi : \mathrm{SL}_2(p) \longrightarrow \mathrm{PSL}_2(p)$$

be the natural projection. By [Hup25, Theorem 8.13], the group $\mathrm{PSL}_2(p)$ contains a subgroup isomorphic to the alternating group A_5. Let $\overline{H} \leq \mathrm{PSL}_2(p)$ be such a subgroup, and set H the full inverse preimage oh $\overline{H}$. Since the kernel of π has order 2, we have

$$|H| = 2|\overline{H}| = 2|A_5| = 120.$$

Moreover,

$$H/\ker(\pi) \simeq \overline{H} \simeq A_5.$$

In particular, H is not solvable, because A_5 is not solvable and every quotient of a solvable group is solvable.

We now show that this group H can occur as a Frobenius complement. Since $H \leq \mathrm{SL}_2(p)$, the group H acts on V, viewed as the elementary abelian additive group of type (p, p), by the natural matrix action. We claim that this action is fixed-point-free.

Let $h \in H$ and suppose that h fixes a nonzero vector $v_1 \in V$. Then 1 is an eigenvalue of h. Since $h \in \mathrm{SL}_2(p)$, we have $\det(h) = 1$, and therefore the other eigenvalue of h is also equal to 1. Hence, for a suitable $v_2 \in V$, the set $\{v_1, v_2\}$ is a basis of V, and with

respect to this basis the matrix of h has the form

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for some $a \in \mathbb{F}_p$. In characteristic p, such a matrix has p-th power equal to the identity. Thus $h^p = 1$. Since $p \nmid |H|$, it follows that $h = 1$.

Consequently, no nonidentity element of H fixes a nonzero element of V. Equivalently,

$$C_V(h) = 0 \qquad \text{for every } h \in H \setminus \{1\}.$$

Therefore H acts fixed-point-freely on V. By Theorem 3.1.1, the semidirect product $G = V \rtimes H$ is a Frobenius group with Frobenius kernel V and Frobenius complement H.

Thompson's theorem establishes that the Frobenius kernel N is always nilpotent, whereas Remark 4.2.8 shows that its nilpotency class may be arbitrarily large. Nevertheless, this class is not completely unrelated to the complement H. The following result illustrates how the prime divisors of $|H|$ can impose strong restrictions on the nilpotency class of the Frobenius kernel N, see [Hup25, Remark 8.8(a)].

Proposition 5.0.3. *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. If 2 divides $|H|$, then H contains a unique involution i. In particular, N is abelian.*

Proof. Since 2 divides $|H|$, by Cauchy's Theorem 1.2.14 there exists an element $i \in H$ of order 2. We first show that i acts on N by inversion. By the fixed-point-free property, the automorphism of N induced by i has no nontrivial fixed points. Hence, by Proposition 1.2.5, the map

$$\mu : N \longrightarrow N, \qquad \mu(n) = n^{-1}n^i,$$

is bijective. Therefore every element of N can be written in the form $n^{-1}n^i$ for some $n \in N$, that is, $N = \{\, n^{-1}n^i \mid n \in N \,\}$.

We now compute the action of i on an element of this form. Since $i^2 = 1$, we have

$$(n^{-1}n^i)^i = n^{-i}n^{i^2} = n^{-i}n^1 = (n^{-1}n^i)^{-1}.$$

Since every element of N is of the form $n^{-1}n^i$, it follows that $x^i = x^{-1}$ for every $x \in N$.

We now prove that i is the unique involution in H. Let $i_1, i_2 \in H$ be two involutions. By the previous argument, both i_1 and i_2 act on N by inversion. Hence, for every $x \in N$, $x^{i_1} = x^{-1} = x^{i_2}$. Therefore $i_2^{-1}i_1$ acts trivially on N. Indeed, since $i_2^{-1} = i_2$, we have

$$x^{i_2^{-1}i_1} = (x^{i_1})^{i_2^{-1}} = (x^{-1})^{i_2} = (x^{i_2})^{-1} = (x^{-1})^{-1} = x.$$

Thus $i_2^{-1}i_1 \in C_G(N)$. Since $i_2^{-1}i_1 \in H$, we obtain $i_2^{-1}i_1 \in H \cap C_G(N)$. The fixed-point-free action of H on N yields $H \cap C_G(N) = 1$. Hence $i_2^{-1}i_1 = 1$, and so $i_1 = i_2$, as claimed. Thus H contains a unique involution.

It remains to show that N is abelian. Let $x, y \in N$. Since i acts on N by automorphisms, we have

$$(xy)^i = x^i y^i = x^{-1}y^{-1}. \tag{5.0.1}$$

On the other hand, since i acts by inversion on every element of N,

$$(xy)^i = (xy)^{-1} = y^{-1}x^{-1}. \tag{5.0.2}$$

Therefore by (5.0.1) and (5.0.2) we obtain

$$x^{-1}y^{-1} = y^{-1}x^{-1}. \tag{5.0.3}$$

Taking inverses on both sides of (5.0.3) gives $yx = xy$. Thus N is abelian, as desired. □

The previous proposition is the first instance of a more general phenomenon. Although Frobenius kernels may have arbitrarily large nilpotency class in general, the prime divisors of the Frobenius complement impose strong restrictions on this class. More precisely, we can recall the following results. If 3 divides $|H|$, then the Frobenius kernel has nilpotency class at most 2 [Bur55]. More generally, if p is an odd prime divisor of $|H|$, then the nilpotency class $cl(N)$ of N satisfies

$$cl(N) \leq \frac{(p-1)^{2^{p-1}-1} - 1}{p-2}. \tag{5.0.4}$$

The estimate in (5.0.4) should be understood as an explicit upper bound, not as the exact value of the best possible bound. Indeed, Higman's theorem on fixed-point-free automorphisms of prime order shows that, for each prime p, there exists a function $h(p)$, the Kreknin–Higman function, depending only on p, such that every finite solvable group admitting a fixed-point-free automorphism of order p is nilpotent and its nilpotency class is at most $h(p)$; see [Hig57] and [HB82, Theorem VIII.10.15]. Moreover, the Kreknin–Higman function satisfies, for odd primes p,

$$h(p) \leq \frac{(p-1)^{2^{p-1}-1} - 1}{p-2}. \tag{5.0.5}$$

In the Frobenius situation this applies as follows. If $p \mid |H|$, then H contains an element h of order p. Since the action of H on N is fixed-point-free, the automorphism of N induced by h is fixed-point-free and has order p. Therefore, Higman's theorem implies

that $cl(N) \leq h(p)$. Combining this with the explicit Kreknin–Higman bound in (5.0.5), we obtain (5.0.4). Among the exact values of $h(p)$ that are known, we recall

$$h(2) = 1, \qquad h(3) = 2, \qquad h(5) = 6.$$

These values follow from the classical results recalled in [Hup25, Remark 8.8(a)]; for the case $p = 5$, see also [Hig57]. Thus the displayed formula gives a general bound, while the optimal value of $h(p)$ can be much smaller. This shows that the nilpotency of the Frobenius kernel is not an isolated property: the fixed-point-free action of the complement has a substantial influence on the internal structure of the kernel.

We conclude with a representation-theoretic property of Frobenius groups. Their special structure is reflected not only in the behaviour of the kernel and the complement, but also in the irreducible characters of the whole group. The next theorem gives a precise description of $\mathrm{Irr}(G)$, the irreducible complex characters of G, when G is a Frobenius group and highlights the relation of $\mathrm{Irr}(G)$ with the irreducible representations of the Frobenius complement.

Theorem 5.0.4. *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H. Let $\mathrm{Irr}(H) = \{\eta_1, \ldots, \eta_{k(H)}\}$ be the set of irreducible complex characters of H, where $k(H)$ denotes the number of conjugacy classes of H. Similarly, let $k(N)$ denote the number of conjugacy classes of N.*

Then the irreducible complex characters of G are exactly the following.

1. *There are $k(H)$ irreducible characters μ_i whose kernels contain N. Each of these irreducible characters μ_i of G is obtained from an irreducible character η_i of H as follows: for $n \in N$ and $h \in H$,*

$$\mu_i(nh) = \eta_i(h), \qquad i = 1, \ldots, k(H).$$

2. *The remaining irreducible characters of G are induced from the nontrivial irreducible characters of N. More precisely, if $\psi \in \mathrm{Irr}(N) \setminus \{1_N\}$, then the induced character ψ^G is an irreducible character of G. Moreover, if $\psi, \varphi \in \mathrm{Irr}(N) \setminus \{1_N\}$, then $\psi^G = \varphi^G$ if and only if there exists $h \in H$ such that $\varphi = \psi^h$. In particular this second family contains*

$$\frac{k(N) - 1}{|H|}$$

distinct irreducible characters of G.

Furthermore, for every $\psi \in \mathrm{Irr}(N) \setminus \{1_N\}$, one has $(\psi^G)_N = \sum_{h \in H} \psi^h$.

The proof of Theorem 5.0.4 relies on some deep results on induced representations, in particular the Mackey decomposition formula [Hup25, Theorem 16.9]. We do not develop these tools in detail, but for the reader's convenience, we give an overview of the main character-theoretic ideas of the proof.

Sketch of proof of Theorem 5.0.4. We first consider the irreducible characters which come from the complement H. Let $\mathrm{Irr}(H) = \{\eta_1, \ldots, \eta_{k(H)}\}$. Since $G = NH$ and $N \cap H = 1$, every element of G can be written uniquely in the form nh, with $n \in N$ and $h \in H$. For each $i = 1, \ldots, k(H)$, define a function $\mu_i : G \to \mathbb{C}$ by

$$\mu_i(nh) = \eta_i(h) \qquad \text{for every } n \in N,\ h \in H.$$

This is a character of G. Indeed, it is afforded by the representation of G in which the normal subgroup N acts trivially and the action of the quotient is given by the representation affording η_i. Moreover, μ_i is irreducible because η_i is irreducible.

For every $n \in N$, we have

$$\mu_i(n) = \mu_i(n \cdot 1) = \eta_i(1) = \mu_i(1),$$

and therefore $N \leq \ker(\mu_i)$.

Thus the characters μ_i are precisely the irreducible characters of G which are obtained from the quotient $G/N \simeq H$.

We now turn to the irreducible characters which come from the kernel. Let

$$\mathrm{Irr}(N) = \{1_N, \psi_2, \ldots, \psi_{k(N)}\}.$$

Since H acts on N by conjugation, every element $h \in H$ induces a permutation of the conjugacy classes of N. If $h \neq 1$, then the only conjugacy class of N fixed by h is the class $\{1\}$. Indeed, suppose that the conjugacy class of an element $n \in N$ is fixed by h. This means that n^h is conjugate to n in N. Since the action of H on N is fixed-point-free, the automorphism $n \mapsto n^h$ is fixed-point-free for every $h \in H \setminus \{1\}$. Hence, by Proposition 1.2.5, we obtain $n = 1$.

Thus every nonidentity element of H fixes exactly one conjugacy class of N, namely the class of the identity. Since the irreducible characters of N form a basis of the space of class functions on N, the same permutation action may be considered on $\mathrm{Irr}(N)$. More explicitly, for $\psi \in \mathrm{Irr}(N)$ and $h \in H$, we write

$$\psi^h(n) = \psi(h^{-1}nh) \qquad \text{for every } n \in N.$$

By Brauer's theorem on automorphisms, the number of irreducible characters of N fixed by h is equal to the number of conjugacy classes of N fixed by h; see [Hup25, Theorem 13.5]. It follows that the only irreducible character of N fixed by h is the trivial character 1_N. Hence the action of H on $\mathrm{Irr}(N) \setminus \{1_N\}$ is semiregular. In particular, every orbit has length $|H|$, and therefore the number of such orbits is

$$\frac{k(N)-1}{|H|}.$$

Let now $\psi_i, \psi_j \in \mathrm{Irr}(N) \setminus \{1_N\}$. We compute the inner product of the corresponding induced characters. By Frobenius reciprocity Proposition 2.1.6,

$$\langle \psi_i^G, \psi_j^G \rangle_G = \langle \psi_i, (\psi_j^G)_N \rangle_N.$$

At this point we use the Mackey decomposition formula [Hup25, Theorem 16.9]. Since $G = NH$, $N \lhd G$, and $N \cap H = 1$, it gives the following restriction formula:

$$(\psi_j^G)_N = \sum_{h \in H} \psi_j^h.$$

Therefore

$$\langle \psi_i^G, \psi_j^G \rangle_G = \left\langle \psi_i, \sum_{h \in H} \psi_j^h \right\rangle_N = \sum_{h \in H} \langle \psi_i, \psi_j^h \rangle_N. \tag{5.0.6}$$

By Theorem 1.1.10, each summand in (5.0.6) is equal to 1 if $\psi_i = \psi_j^h$, and is equal to 0 otherwise. Moreover, because the action of H on $\mathrm{Irr}(N) \setminus \{1_N\}$ is semiregular, there is at most one element $h \in H$ such that $\psi_i = \psi_j^h$. Hence

$$\langle \psi_i^G, \psi_j^G \rangle_G = \begin{cases} 1, & \text{if } \psi_i = \psi_j^h \text{ for some } h \in H, \\ 0, & \text{otherwise.} \end{cases}$$

Taking $i = j$, we obtain

$$\langle \psi_i^G, \psi_i^G \rangle_G = 1.$$

Thus ψ_i^G is irreducible. Moreover, two induced characters ψ_i^G and ψ_j^G are equal if and only if ψ_i and ψ_j lie in the same H-orbit.

It remains to observe that these induced characters are distinct from the characters μ_i obtained from H. Let $\psi \in \mathrm{Irr}(N) \setminus \{1_N\}$. By Frobenius reciprocity Proposition 2.1.6,

$$\langle \psi^G, \mu_i \rangle_G = \langle \psi, (\mu_i)_N \rangle_N.$$

Since $N \leq \ker(\mu_i)$, the restriction of μ_i to N is $(\mu_i)_N = \mu_i(1)1_N$. Thus

$$\langle \psi^G, \mu_i \rangle_G = \mu_i(1)\langle \psi, 1_N \rangle_N = 0,$$

because $\psi \neq 1_N$. Hence the two families of characters are disjoint.

Finally, we show that no irreducible characters of G have been missed. The sum of the squares of the degrees of the characters μ_i is

$$\sum_{i=1}^{k(H)} \mu_i(1)^2 = \sum_{i=1}^{k(H)} \eta_i(1)^2 = |H|. \tag{5.0.7}$$

For the second family, choose one representative from each H-orbit on $\mathrm{Irr}(N) \setminus \{1_N\}$. Since each orbit has length $|H|$, the sum of the squares of the degrees of the induced characters can be written as

$$\frac{1}{|H|} \sum_{\psi \in \mathrm{Irr}(N) \setminus \{1_N\}} (\psi^G(1))^2.$$

By Lemma 2.1.5, for every $\psi \in \mathrm{Irr}(N)$ we have

$$\psi^G(1) = [G : N]\psi(1) = |H|\psi(1).$$

Therefore

$$\frac{1}{|H|} \sum_{\psi \neq 1_N} (\psi^G(1))^2 = \frac{1}{|H|} \sum_{\psi \neq 1_N} |H|^2 \psi(1)^2 = |H| \sum_{\psi \neq 1_N} \psi(1)^2. \tag{5.0.8}$$

Since

$$\sum_{\psi \in \mathrm{Irr}(N)} \psi(1)^2 = |N|$$

and $1_N(1) = 1$, (5.0.8) becomes

$$\frac{1}{|H|} \sum_{\psi \neq 1_N} (\psi^G(1))^2 = |H|(|N| - 1). \tag{5.0.9}$$

Summing (5.0.7) and (5.0.9), we obtain

$$\sum_{i=1}^{k(H)} \mu_i(1)^2 + \frac{1}{|H|} \sum_{\psi \neq 1_N} (\psi^G(1))^2 = |H| + |H|(|N| - 1) = |H||N| = |G|.$$

By Corollary 1.1.17, this proves that the characters listed in the theorem exhaust all irre-

ducible characters of G. $\square$

We conclude this chapter by stating the general description for the irreducible representations of G over any algebraically closed field whose characteristic does not divide the order of the group G. The proof of this more general formulation requires additional results from modular representation theory and will not be included in this thesis. The interested reader can find it, for example, in [Hup25, Theorem 16.13].

Here $\mathrm{Irr}_K(X)$ denotes the set of irreducible characters of X over K.

Theorem 5.0.5. *Let G be a Frobenius group with Frobenius kernel N and Frobenius complement H, and let K be an algebraically closed field such that $\mathrm{char}\, K \nmid |G|$. Then the irreducible K-characters of G are exactly the following.*

1. *There are $k(H)$ irreducible characters μ_i whose kernels contain N. If $\mathrm{Irr}_K(H) = \{\eta_1, \ldots, \eta_{k(H)}\}$, then these characters are defined by*

$$\mu_i(nh) = \eta_i(h) \qquad \text{for every } n \in N,\ h \in H.$$

2. *There are*

$$\frac{k(N) - 1}{|H|}$$

irreducible characters of the form ψ^G, where ψ runs through a set of representatives of the H-orbits on $\mathrm{Irr}_K(N) \setminus \{1_N\}$. Moreover, $\psi_i^G = \psi_j^G$ if and only if $\psi_i = \psi_j^h$ for some $h \in H$, and

$$(\psi_i^G)_N = \sum_{h \in H} \psi_i^h.$$

In addition, if V is an irreducible KG-module on which N acts nontrivially, then the restriction V_H is a direct sum of regular KH-modules. In particular, V_H contains a submodule affording the trivial representation of H.

Bibliography

[Bur55] William Burnside. *Theory of Groups of Finite Order*. Dover Publications, New York, 2 edition, 1955.

[Fro01] Georg Frobenius. Über auflösbare gruppen. iv. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, pages 1216–1230, 1901. JFM 32.0137.01.

[Gor80] Daniel Gorenstein. *Finite Groups*. Chelsea Publishing Company, New York, 2nd edition, 1980.

[HB82] Bertram Huppert and Norman Blackburn. *Finite Groups III*, volume 243 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin–Heidelberg, 1982.

[Hig57] Graham Higman. Groups and rings having automorphisms without non-trivial fixed elements. *Journal of the London Mathematical Society*, 32:321–334, 1957.

[Hup25] Bertram Huppert. *Finite Groups I*, volume 364 of *Grundlehren der mathematischen Wissenschaften*. Springer Nature Switzerland, Cham, 2025. Translated by Christopher A. Schroeder from the German edition *Endliche Gruppen I*, Springer-Verlag, Berlin Heidelberg, 1967.

[Isa94] I. Martin Isaacs. *Algebra: A Graduate Course*. Wadsworth, Belmont, California, 1994.

[Isa08] I. Martin Isaacs. *Finite Group Theory*, volume 92 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2008.

[JL04] Gordon James and Martin Liebeck. *Representations and Characters of Groups*. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 2 edition, 2004.

[Ros94] John S. Rose. *A Course on Group Theory*. Dover Publications, Mineola, New York, 1994. Unabridged and unaltered republication of the work first published by Cambridge University Press, Cambridge, 1978; reissued in 2012.

[Ros09] H. E. Rose. *A Course on Finite Groups*. Universitext. Springer, London, 2009.

[Tho59] John G. Thompson. Finite groups with fixed-point-free automorphisms of prime order. *Proceedings of the National Academy of Sciences of the United States of America*, 45(4):578–581, 1959.

[Tho64] John G. Thompson. Normal p-complements for finite groups. *Journal of Algebra*, 1(1):43–46, 1964.